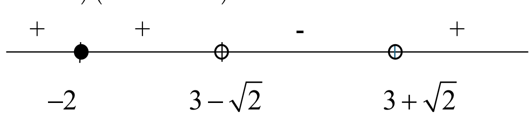
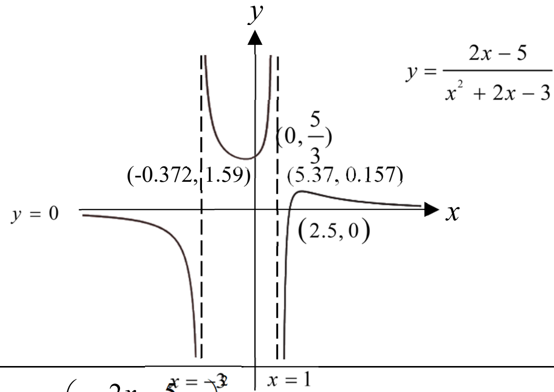
Q1	Suggested Answers

(a)	$y=\frac{a}{x^2}+be^{2x}+c$
	Sub $x=1, y=2e^2-1$
	$a+be^2+c=2e^2-1$ ---(1)
	
	After scaling,
	$y=\frac{a}{\left(\frac{x}{2}\right)^2}+be^{2\left(\frac{x}{2}\right)}+c=\frac{4a}{x^2}+be^x+c$
	Sub $x=-1, y=8+\frac{2}{e}$
	$4a+be^{-1}+c=8+\frac{2}{e}$ ---(2)
	
	After translation
	$y=\frac{a}{x^2}+be^{2x}+c-1$
	Sub
	$x=\sqrt{3}, y=2e^{2\sqrt{3}}-4$
	$\frac{a}{3}+be^{2\sqrt{3}}+c=2e^{2\sqrt{3}}-3$---(3)
	
	Using GC,
	$a=3, b=2, c=-4$

Q2	**Suggested Answers**
(a)	**<u>Method 1:</u>** $x^2+4x+9=(x+2)^2+5$ Since $(x+2)^2 \geq 0, (x+2)^2+5>0$. Therefore, x^2+4x+9 is always positive for all real values of x.
	<u>Method 2:</u> $b^2-4ac=4^2-4(1)(9)=-20$ Since the discriminant <0 and that the coefficient of x^2 is positive, therefore, x^2+4x+9 is always positive for all real values of x.
(b)	$\dfrac{(x^2+4x+9)(x+2)^2}{x^2-6x+7} \leq 0$ Since $x^2+4x+9>0$ $\dfrac{(x+2)^2}{x^2-6x+7} \leq 0$ $\dfrac{(x+2)^2}{(x-3)^2-2} \leq 0$ $\dfrac{(x+2)^2}{(x-3+\sqrt{2})(x-3-\sqrt{2})} \leq 0$  $\therefore x=-2$ or $3-\sqrt{2}<x<3+\sqrt{2}$

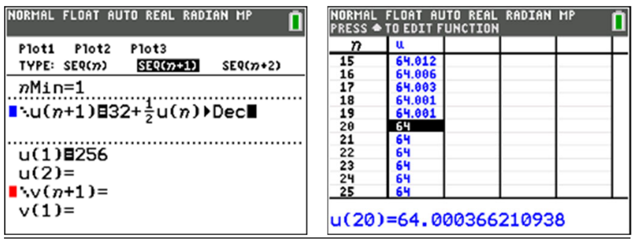
Q3	Suggested Answers
(a)	
(b)	$(x+3)^2+\left(\frac{2x-5}{x^2+2x-3}\right)^2=k^2$ $(x+3)^2+y^2=k^2$ $k=\sqrt{\left(\frac{5}{3}-0\right)^2+\left(0-(-3)\right)^2}$ $=\sqrt{\frac{25}{9}+9}$ $=\sqrt{\frac{106}{9}}$ $=\frac{\sqrt{106}}{3}$ $\therefore k>\frac{\sqrt{106}}{3}$

Q4	Suggested Answers
(a)	$U_4=p+qU_3\Rightarrow 76=p+88q---(1)$ $U_5=p+qU_4\Rightarrow 70=p+76q---(2)$ Solving equation (1) and (2): $p=32, q=0.5$
(b)	**<u>Method 1:</u>** $U_3=p+qU_2\Rightarrow 88=32+(0.5)U_2\Rightarrow U_2=112$ $U_2=p+qU_1\Rightarrow 112=32+(0.5)U_1\Rightarrow U_1=160$ $U_1=p+qU_0\Rightarrow 160=32+(0.5)U_0\Rightarrow U_0=256$

(b) **Method 2:**

$$U_3 = 32 + \tfrac{1}{2}U_2$$
$$= 32 + \tfrac{1}{2}\left(32 + \tfrac{1}{2}U_1\right)$$
$$= 32 + \tfrac{1}{2}\left(32 + \tfrac{1}{2}\left(32 + \tfrac{1}{2}U_0\right)\right)$$
$$88 = 32 + \tfrac{1}{2}\left(32 + \tfrac{1}{2}\left(32 + \tfrac{1}{2}U_0\right)\right)$$
$$\therefore U_0 = 256$$

(c) **Method 1: Using GC to get 64**



Method 2:

As $t \to \infty, C_t \to L, C_{t+1} \to L$

$$L = 32 + \frac{1}{2}L$$
$$2L = 64 + L$$
$$L = 64$$

The readings **decreases** and **converges** to 64.

Q5	Suggested Answers

(a)

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}3\\2\\-1\end{pmatrix}\times\begin{pmatrix}4\\-3\\2\end{pmatrix} = \begin{pmatrix}1\\-10\\-17\end{pmatrix}$$

Area of ΔAOB

$$= \frac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|$$
$$= \frac{1}{2}\left|\begin{pmatrix}1\\-10\\-17\end{pmatrix}\right|$$
$$= \frac{1}{2}\sqrt{1+100+289}$$
$$= \frac{1}{2}\sqrt{390}\text{ units}^2$$

(b)	**Method 1:** Perpendicular ht of tetrahedron = $\frac{1}{\sqrt{1+100+289}}\left\lvert \overrightarrow{OC}\cdot\begin{pmatrix}1\\-10\\-17\end{pmatrix}\right\rvert = \frac{1}{\sqrt{390}}\left\lvert\begin{pmatrix}3\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-10\\-17\end{pmatrix}\right\rvert = \frac{30}{\sqrt{390}}$ Vol of tetrahedron OABC = $\frac{1}{3}\left(\frac{\sqrt{390}}{2}\right)\left(\frac{30}{\sqrt{390}}\right) = 5\text{ units}^3$
	Method 2 Let *X* be the foot of perpendicular from *C* to plane *OAB*. $\overrightarrow{OX} = \begin{pmatrix}3\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\-10\\-17\end{pmatrix}$, for some $\mu \in \mathbb{R}$ Equation of plane *OAB*: $\mathbf{r}\cdot\begin{pmatrix}1\\-10\\-17\end{pmatrix} = 0$ Since the point *X* also lies on the plane *OAB*, $\left(\begin{pmatrix}3\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\-10\\-17\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-10\\-17\end{pmatrix} = 0$ $3+\mu+10+100\mu+17+289\mu = 0$ $390\mu = -30$ $\mu = -\frac{1}{13}$ $\overrightarrow{CX} = \overrightarrow{OX} - \overrightarrow{OC}$ $= \begin{pmatrix}3\\-1\\-1\end{pmatrix} - \frac{1}{13}\begin{pmatrix}1\\-10\\-17\end{pmatrix} - \begin{pmatrix}3\\-1\\-1\end{pmatrix}$ $= -\frac{1}{13}\begin{pmatrix}1\\-10\\-17\end{pmatrix}$ perpendicular height, $\lvert\overrightarrow{CX}\rvert = \frac{1}{13}\sqrt{1+(-10)^2+(-17)^2}$ $= \frac{\sqrt{390}}{13}$ $= \frac{30}{\sqrt{390}}$

(c)	**Method 1** Perpendicular distance from B to line OA $= \lvert \mathbf{b} \times \hat{\mathbf{a}} \rvert$ $= \dfrac{\sqrt{390}}{\sqrt{9+4+1}}$ $= \sqrt{\dfrac{195}{7}}$ **or** 5.28
	Method 2 Area of OAB = $\dfrac{1}{2}\sqrt{390} = \dfrac{1}{2}OA(h)$ Thus, perpendicular distance from B to line $OA = h = \sqrt{\dfrac{390}{14}} = \sqrt{\dfrac{195}{7}}$

Q6	Suggested Answers
(a)	**NEW** $S_n = n^2 + 2n$ $S_n = n^2 + 2n$ $u_n = S_n - S_{n-1}$ $= (n^2 + 2n) - ((n-1)^2 + 2(n-1))$ $= n^2 + 2n - n^2 + 2n - 1 - 2n + 2$ $= 2n + 1$ $u_{n+1} - u_n = (2(n+1)+1) - (2n+1)$ $= 2n + 3 - 2n - 1$ $= 2$ Since $u_{n+1} - u_n = 2$ is a constant, it is an arithmetic series. $\therefore d = 2$ $u_n = 2n + 1$ $a + (n-1)2 = 2n + 1$ $a + 2n - 2 = 2n + 1$ $a = 3$
(b)	$a = 12$ $u_6 = -\dfrac{3}{8}$ $\Rightarrow ar^5 = -\dfrac{3}{8}$ $\Rightarrow r^5 = -\dfrac{1}{32}$ $\Rightarrow r = -\dfrac{1}{2}$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$= \frac{12\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)}$$

$$= 8\left(1-\left(-\frac{1}{2}\right)^n\right)$$

$$S = \frac{a}{1-r}$$

$$= \frac{12}{1-\left(-\frac{1}{2}\right)}$$

$$= 8$$

$$\left|S_n - S\right| < 0.001$$

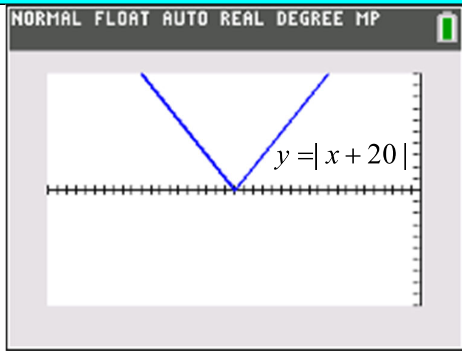
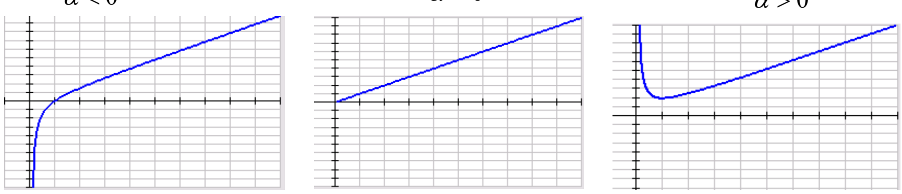
$$\left|8\left(1-\left(-\frac{1}{2}\right)^n\right)-8\right| < 0.001$$

$$\left|8\left(1-\left(-\frac{1}{2}\right)^n\right)-8\right| - 0.001 < 0$$

From GC,

n	$\left\|8\left(1-\left(-\frac{1}{2}\right)^n\right)-8\right\| - 0.001$
12	$9.53\times10^{-4} > 0$
13	$-2.34\times10^{-5} < 0$

Therefore, the least value of n is 13.

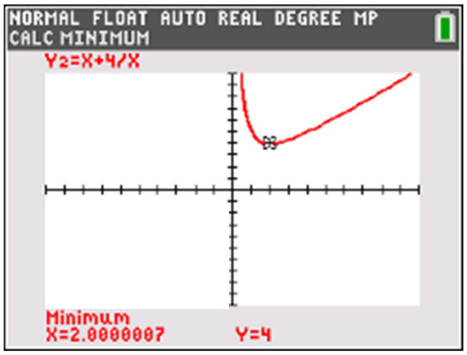
Q7	Suggested Answers
(a)	 $\mathrm{R_f} = [0, \infty)$ $\mathrm{D_g} = \mathbb{R}^+$ $\mathrm{R_f} \subsetneq \mathrm{D_g}$ Thus gf does not exist.
(b)	**Method 1:** Consider the different scenarios: $\alpha < 0$ $\qquad$ $\alpha = 0$ $\qquad$ $\alpha > 0$  If $\alpha > 0$, g will have a turning point at $x = \sqrt{\alpha}$, making g not an one-one function and g^{-1} will not exist. Hence, $\alpha \leq 0$. **Method 2:** We need $\mathrm{g}'(x) = 1 - \frac{\alpha}{x^2} > 0$ for g^{-1} to exist. Since $x^2 > 0, \alpha < 0$ Also, when $\alpha = 0$, $\mathrm{g}(x) = x$ is also an one-one function. Hence, $\alpha \leq 0$

(c)

$$\mathrm{g}(x) = x + \frac{4}{x}$$

$$\mathrm{g}'(x) = 1 - \frac{4}{x^2} = 0$$

$$x = \pm 2$$



From graph, least $\beta = 2$ since we need g to be one-one function.

(or since $x > 0$, $\beta = 2$)

$$y = x + \frac{4}{x}$$

$$xy = x^2 + 4$$

$$x = \frac{y \pm \sqrt{y^2 - 4(1)(4)}}{2}$$

since $x \geq 2, x = \dfrac{y + \sqrt{y^2 - 16}}{2}$

$$\mathrm{g}^{-1}(x) = \frac{1}{2}(x + \sqrt{x^2 - 16})$$

$$\mathrm{D}_{\mathrm{g}^{-1}} = \mathrm{R}_{\mathrm{g}} = [4, \infty)$$

Q8 **Suggested Answers**

(a)

$$y = \sqrt{\ln(x + \mathrm{e})} \Rightarrow y^2 = \ln(x + \mathrm{e}) \Rightarrow \mathrm{e}^{y^2} = x + \mathrm{e}$$

$$2y\mathrm{e}^{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-y^2} \text{ (shown)}$$

OR

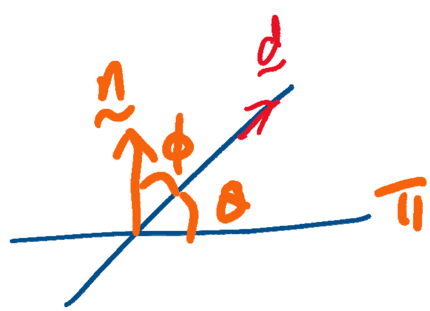
Differentiate w.r.t. *x* gives

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x + \mathrm{e}} = \frac{1}{\mathrm{e}^{y^2}} \text{ since } y^2 = \ln(x + \mathrm{e}) \Rightarrow x + \mathrm{e} = \mathrm{e}^{y^2}$$

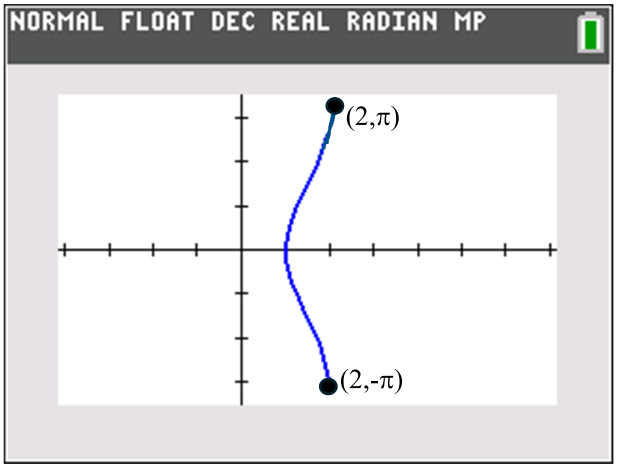
That is, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-y^2}$.

Differentiate w.r.t. *x* gives

	$2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=-2y\frac{dy}{dx}e^{-y^2}$ $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=-y\frac{dy}{dx}e^{-y^2}$ When $x=0$, $y=1$, $\frac{dy}{dx}=\frac{1}{2e}$, $\frac{d^2y}{dx^2}=-\frac{3}{4e^2}$. $y=1+\frac{1}{2e}x+\frac{1}{2!}\left(-\frac{3}{4e^2}\right)x^2+\ldots=1+\frac{1}{2e}x-\frac{3}{8e^2}x^2+\ldots$.
(b)	$\ln(x+e)=\ln e\left(1+\frac{x}{e}\right)$ $=1+\ln\left(1+\frac{x}{e}\right)$ $=1+\frac{x}{e}-\frac{1}{2}\left(\frac{x}{e}\right)^2+\ldots$ $=1+\left(\frac{x}{e}-\frac{x^2}{2e^2}+\ldots\right)$ $y=\sqrt{\ln(x+e)}$ $=\left[\ln(x+e)\right]^{\frac{1}{2}}$ $=\left[1+\left(\frac{x}{e}-\frac{x^2}{2e^2}+\ldots\right)\right]^{\frac{1}{2}}$ $=1+\frac{1}{2}\left(\frac{x}{e}-\frac{x^2}{2e^2}+\ldots\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{e}-\frac{x^2}{2e^2}+\ldots\right)^2+\ldots$ $=1+\frac{1}{2e}x-\frac{3}{8e^2}x^2+\ldots$ which agrees with the expansion in (a).
(c)	$\sqrt{\ln\left(\frac{1+10e}{10}\right)}=\sqrt{\ln\left(\frac{1}{10}+e\right)}$ $\approx 1+\frac{1}{2e}\left(\frac{1}{10}\right)-\frac{3}{8e^2}\left(\frac{1}{10}\right)^2$ $=1+\frac{1}{20e}-\frac{3}{800e^2}$ $=\frac{800e^2+40e-3}{800e^2}$

Q9	Suggested Answers
(a)	$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\cos\theta = \dfrac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}}{\sqrt{1+4+1}\sqrt{1+16+25}} = \dfrac{14}{\sqrt{6}\sqrt{42}}$ $\theta = 28.1^o$
(b)	$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 4$ $\Pi_1 : x - 2y + z = 4$ $\Pi_2 : x - 4y + 5z = 12$ Using GC, Line of intersection has equation $\mathbf{r} = \begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$
(c)	$l : \mathbf{r} = \begin{pmatrix} m \\ 2m+1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3n \\ -3 \\ n \end{pmatrix}$ $\theta = \sin^{-1}\dfrac{2}{\sqrt{6}}$ $\sin\theta = \dfrac{2}{\sqrt{6}}$ $\cos(\dfrac{\pi}{2} - \theta) = \dfrac{2}{\sqrt{6}}$ $\cos\phi = \dfrac{2}{\sqrt{6}}$ 

	$\dfrac{\begin{pmatrix}3n\\-3\\n\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\sqrt{(3n)^2+9+n^2}\sqrt{6}}=\dfrac{2}{\sqrt{6}}$ $\dfrac{3n+6+n}{\sqrt{10n^2+9}}=2$ $(2n+3)^2=10n^2+9$ $n=0$ (rej, n>0) or $n=2$ **Using distance of point to plane formula** $\left\|\dfrac{\mathbf{b}\bullet\mathbf{n}-D}{\|\mathbf{n}\|}\right\|$ Perpendicular distance from A to plane $\Pi_1=\dfrac{15}{\sqrt{6}}$ $\dfrac{\left\|\begin{pmatrix}m\\2m+1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}-4\right\|}{\sqrt{6}}=\dfrac{15}{\sqrt{6}}$ $\dfrac{\|m-4m-2-3-4\|}{\sqrt{6}}=\dfrac{15}{\sqrt{6}}$ $-3m-9=\pm 15$ $m=-8$ (rej, m>0) or $m=2$

Q10	Suggested Answers
(a)	
(b)	$x = 1 + t^2$, $y = 2\sin^{-1} t$ for $\lvert t \rvert \le 1$. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$ $= \frac{\frac{2}{\sqrt{1-t^2}}}{2t}$ $= \frac{1}{t\sqrt{1-t^2}}$
(c)	At P, $y = 0 \Rightarrow 2\sin^{-1} t = 0 \Rightarrow t = 0$. When $t = 0$, $x = 1$ which gives the x-coordinate of P. Since $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{x=1} = \left.\frac{1}{t\sqrt{1-t^2}}\right\|_{t=0}$ is undefined, the robot is moving in a direction parallel to the y-axis. So the equation of the line is $x = 1$.
(d)	$y = \frac{\pi}{3} \Rightarrow 2\sin^{-1} t = \frac{\pi}{3} \Rightarrow \sin^{-1} t = \frac{\pi}{6} \Rightarrow t = \frac{1}{2}$. $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{t=\frac{1}{2}} = \frac{1}{\frac{1}{2}\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{4}{\sqrt{3}}$. The angle θ which the direction of motion of the robot makes with the positive x-axis is given by

	$\tan\theta = \dfrac{4}{\sqrt{3}} \Rightarrow \theta \approx 66.6^\circ$. (or 1.16 rad)
(e)	$s = \sqrt{\left(1+t^2-2\right)^2 + \left(2\sin^{-1}t - 2\right)^2}$ $= \sqrt{\left(t^2-1\right)^2 + \left[2\left(\sin^{-1}t-1\right)\right]^2}$ $= \sqrt{\left(t^2-1\right)^2 + 4\left(\sin^{-1}t-1\right)^2}$
(f)	$s^2 = \left(t^2-1\right)^2 + 4\left(\sin^{-1}t-1\right)^2$. Differentiate w.r.t. t gives $s^2 = \left(t^2-1\right)^2 + 4\left(\sin^{-1}t-1\right)^2$ $2s\dfrac{\mathrm{d}s}{\mathrm{d}t} = 4t\left(t^2-1\right) + \dfrac{8\left(\sin^{-1}t-1\right)}{\sqrt{1-t^2}}$ For least s, $\dfrac{\mathrm{d}s}{\mathrm{d}t} = 0$ $\Rightarrow 4t\left(t^2-1\right) + \dfrac{8\left(\sin^{-1}t-1\right)}{\sqrt{1-t^2}} = 0$ $\Rightarrow t\left(t^2-1\right) + \dfrac{2\left(\sin^{-1}t-1\right)}{\sqrt{1-t^2}} = 0$ By GC, $t = 0.86879$. Minimum distance $s = \sqrt{\left(0.86879^2-1\right)^2 + 4\left(\sin^{-1}0.86879-1\right)^2} \approx 0.267$ Since $s > 0.25$, the robot will not be attracted by the magnet.