2015 SRJC H2 Mathematics Prelim Paper 2

Section A: Pure Mathematics [40 marks]

1 Functions f and g are defined as below.

$$\mathrm{f}: x \mapsto \frac{1}{x-1}, \qquad x \geq 2$$
$$\mathrm{g}: x \mapsto \ln(2-x)-x, \quad x < -2$$

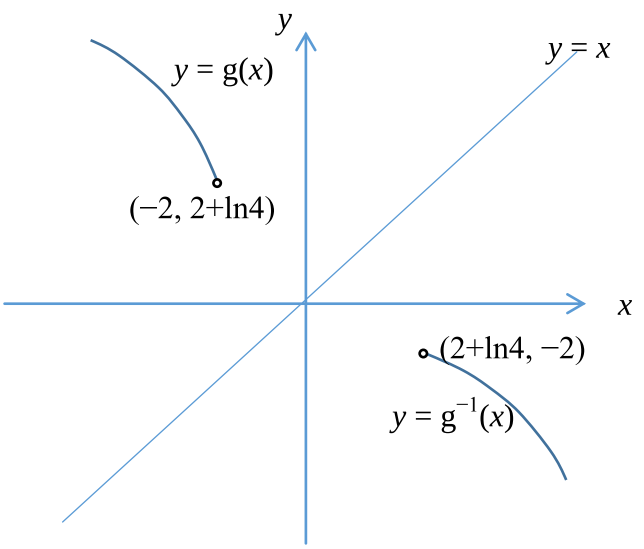
(i) Sketch the graph of $y = \mathrm{g}(x)$ and state its exact range. [3]

(ii) Determine whether the composite functions fg and gf exist, justifying your answer. Find the range of the composite function if it exists. [4]

(iii) On the same diagram as part **(i)**, sketch the graph of $y = \mathrm{g}^{-1}(x)$, indicating on your sketch the line in which the graph of $y = \mathrm{g}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{g}^{-1}(x)$. [2]

Suggested Solution

(i)

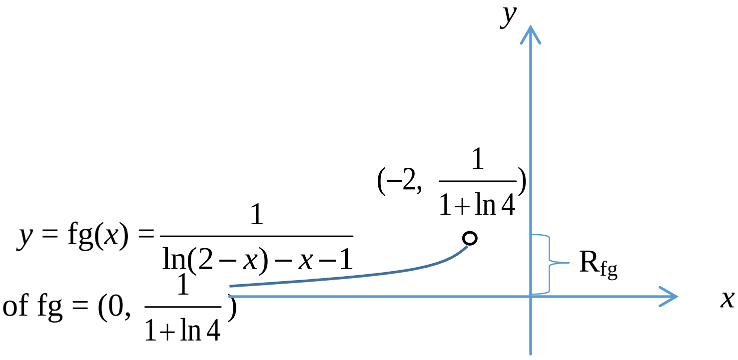


Range of g, $R_g = (2 + \ln 4, +\infty)$

(ii) From GC, the range of f is (0, 1]. Since $R_f \not\subset D_g = (-\infty, -2)$, the composite gf does not exist.

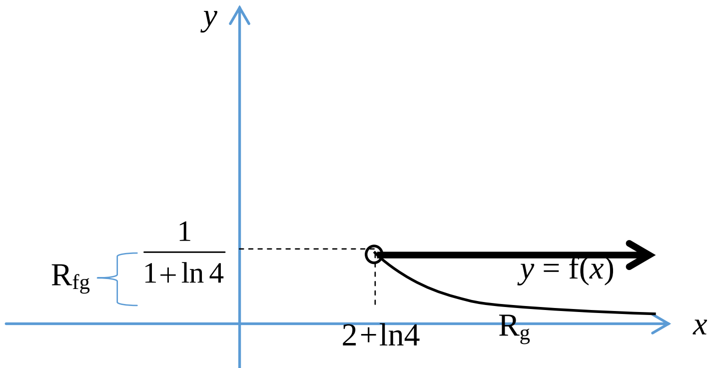
Since $R_g = (2 + \ln 4, +\infty) \subset [2, \infty) = D_f$ the composite fg exists.

<u>Direct Method for finding range of fg</u>



Range of fg $= (0, \frac{1}{1+\ln 4})$

<u>Indirect Method for finding range of fg</u>



Range of fg = $(0, \frac{1}{1+\ln 4})$

2 A tank contains water which is heated by an electric water heater working under the action of a thermostat. When the water heater is first switched on, the temperature of the water is 35°C. The heater causes the temperature to increase at a rate $r^\circ\text{C}$ per minute, where *r* is a constant, until the water temperature hits 75°C.The heater then switches off.

(i) Write down, in terms of *r*, the time taken for the temperature to increase from 35°C to 75°C. [1]

The temperature of the water then immediately starts to decrease. The temperature of the water at time *t* minutes after the heater is switched off is $\theta^\circ\text{C}$. It is known that the temperature of the water decreases at a variable rate $k(\theta - 25)^\circ\text{C}$ per minute, where *k* is a positive constant, until $\theta = 35$.

(ii) Write down a differential equation involving θ and *t*, to represent the situation as the temperature is decreasing. [1]

(iii) Given that when $\theta = 55$, the temperature is decreasing at a rate of 5°C per minute, find the total length of time for the temperature to increase from 35°C to 75°C and then decrease to 35°C, leaving your answer in exact form, in terms of *r*. [7]

Suggested Solution

(i) Time taken $= \dfrac{40}{r}$ mins

(ii) $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 25)$

(iii) $\displaystyle\int \frac{1}{\theta - 25}\ \mathrm{d}\theta = -\int k\ \mathrm{d}t$

$\ln(\theta - 25) = -kt + c$, where c is a constant, since $\neg > 25^{\circ}$C

$(\theta - 25) = \mathrm{e}^{-kt}.\mathrm{e}^{c}$

$\theta = 25 + A\mathrm{e}^{-kt}$, where A is an arbitrary constant.

When $\theta = 55$, $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -5$, $k = \dfrac{1}{6}$

When $t = 0$ and $\theta = 75$, $A = 50$

$\therefore \theta = 25 + 50\mathrm{e}^{-\frac{1}{6}t}$

When $\theta = 35$, $35 = 25 + 50\mathrm{e}^{-\frac{1}{6}t}$

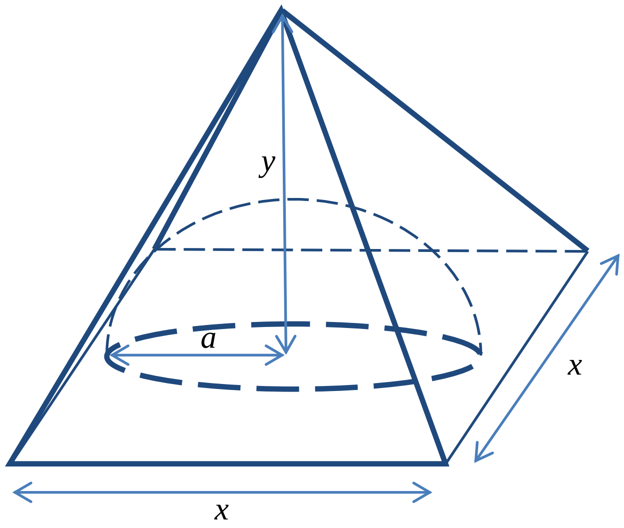
$\Rightarrow \mathrm{e}^{-\frac{1}{6}t} = \dfrac{1}{5}$

$\Rightarrow t = -6\ln\left(\frac{1}{5}\right) = 6\ln 5$

Total length of time = $\left(\dfrac{40}{r} + 6\ln 5\right)$ mins

3 [It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a right square pyramid with a square base of length x and height h is $\frac{1}{3}x^2 h$.]

In the diagram below, a hemisphere of fixed radius a cm lies on the base of a right pyramid such that its curved surface is in contact with all four faces of the pyramid.



The pyramid has a square base of length x cm and height y cm. Given that the volume of region inside the pyramid that is not part of the hemisphere is denoted by V,

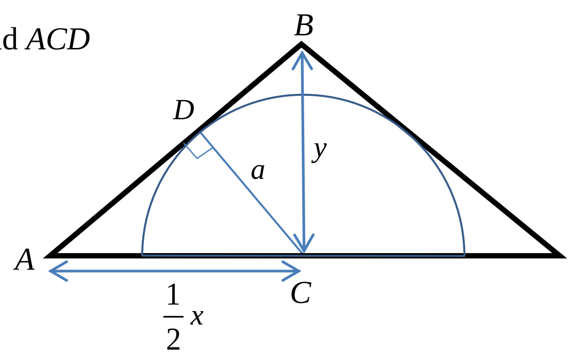
(i) show that $V = \frac{1}{3}\frac{ax^3}{\sqrt{x^2 - 4a^2}} - \frac{2}{3}\pi a^3$. [3]

(ii) use differentiation to find, in terms of a, the minimum value of V exactly, proving that it is a minimum. [7]

Suggested Solution

(i) **Method 1**

Consider the similar triangles ABC and ACD



$$\frac{BC}{CD} = \frac{AB}{AC}$$

$$\frac{y}{a} = \frac{\sqrt{\left(\frac{1}{2}x\right)^2 + y^2}}{\frac{1}{2}x}$$

$$\frac{y}{a} = \frac{\sqrt{\frac{1}{4}x^2 + y^2}}{\frac{1}{2}x}$$

$$\frac{y}{a}=\frac{\frac{1}{2}\sqrt{x^2+4y^2}}{\frac{1}{2}x}$$

$$xy = a\sqrt{x^2+4y^2}$$

$$x^2y^2 = a^2x^2 + 4a^2y^2$$

$$y^2(x^2-4a^2) = a^2x^2$$

$$y^2=\frac{a^2x^2}{x^2-4a^2}$$

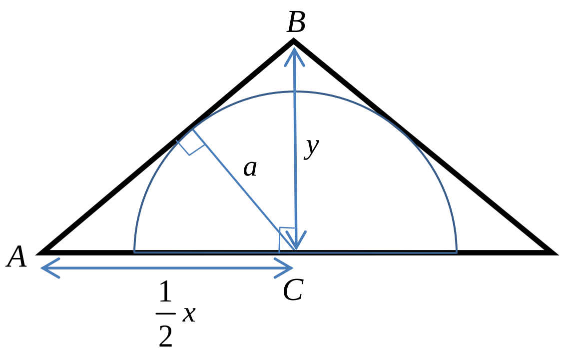
$$y=\frac{ax}{\sqrt{x^2-4a^2}} \text{ or } y=-\frac{ax}{\sqrt{x^2-4a^2}} \text{ (rejected } \because x>0)$$

$$\therefore V=\frac{1}{3}x^2y-\frac{2}{3}\pi a^3$$

$$=\frac{1}{3}\frac{ax^3}{\sqrt{x^2-4a^2}}-\frac{2}{3}\pi a^3$$

Method 2

Considering the area of ΔABC, we have



$$\frac{1}{2}\left(\frac{x}{2}\right)y=\frac{1}{2}\left(\sqrt{\left(\frac{x}{2}\right)^2+y^2}\right)a$$

$$\left(\frac{x^2}{4}\right)y^2=\left(\frac{x^2}{4}+y^2\right)a^2$$

$$\left(\frac{x^2}{4}-a^2\right)y^2=a^2\frac{x^2}{4}$$

$$y=\sqrt{\frac{a^2x^2}{x^2-4a^2}}=\frac{ax}{\sqrt{x^2-4a^2}}$$

$$\therefore V=\frac{1}{3}x^2y-\frac{2}{3}\pi a^3$$

$$=\frac{1}{3}\frac{ax^3}{\sqrt{x^2-4a^2}}-\frac{2}{3}\pi a^3$$

(ii) $V=\frac{1}{3}\frac{ax^3}{\sqrt{x^2-4a^2}}-\frac{2}{3}\pi a^3$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{1}{3}a\frac{\sqrt{x^2-4a^2}\left(3x^2\right)-\frac{2x}{2\sqrt{x^2-4a^2}}x^3}{\left(\sqrt{x^2-4a^2}\right)^2}$$

$$=\frac{1}{3}a\frac{\left(x^2-4a^2\right)\left(3x^2\right)-x^4}{\left(x^2-4a^2\right)^{\frac{3}{2}}}$$

$$=\frac{1}{3}a\frac{2x^4-12a^2x^2}{\left(x^2-4a^2\right)^{\frac{3}{2}}}$$

$$=\frac{2}{3}a\frac{x^2\left(x^2-6a^2\right)}{\left(x^2-4a^2\right)^{\frac{3}{2}}}$$

Let $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$.

$$\therefore\ \frac{2}{3}a\frac{x^2\left(x^2-6a^2\right)}{\left(x^2-4a^2\right)^{\frac{3}{2}}} = 0$$

$\Rightarrow x^2 - 6a^2 = 0$ or $x = 0$ (Reject $\because x > 0$)

$\Rightarrow x = \sqrt{6}\,a$ or $x = -\sqrt{6}\,a$ (Reject $\because x > 0$)

<u>Method 1</u>

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{3}a\frac{x^4-6a^2x^2}{\left(x^2-4a^2\right)^{\frac{3}{2}}}\right)$$

$$=\frac{2}{3}a\frac{\left(x^2-4a^2\right)^{\frac{3}{2}}\left(4x^3-12a^2x\right)-\frac{3}{2}(2x)\left(x^2-4a^2\right)^{\frac{1}{2}}\left(x^4-6a^2x^2\right)}{\left(x^2-4a^2\right)^3}$$

When $x = \sqrt{6}\,a$,

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2}{3}a\frac{\left(6a^2-4a^2\right)^{\frac{3}{2}}\left(24\sqrt{6}a^3-12\sqrt{6}a^3\right)-3\sqrt{6}a\left(6a^2-4a^2\right)^{\frac{1}{2}}\left(36a^4-36a^4\right)}{\left(6a^2-4a^2\right)^3}$$

$$=\frac{2}{3}a\frac{48\sqrt{3}a^6-0}{8a^6} = 4\sqrt{3}a > 0$$

Method 2

x	$\sqrt{6}\,a^-$	$\sqrt{6}\,a$	$\sqrt{6}\,a^+$
$\frac{\mathrm{d}V}{\mathrm{d}x}$	-ve since $\frac{x^2\left(x^2-6a^2\right)}{\left(x^2-4a^2\right)^{\frac{3}{2}}}<0$ for $x<\sqrt{6}\,a$	0	+ve since $\frac{x^2\left(x^2-6a^2\right)}{\left(x^2-4a^2\right)^{\frac{3}{2}}}>0$ for $x>\sqrt{6}\,a$
Slope	\	–	/

Hence, V is a minimum value when $x = \sqrt{6}\,a$.

The minimum value of V $= \frac{1}{3}\frac{6\sqrt{6}a^4}{\sqrt{6a^2-4a^2}}-\frac{2}{3}\pi a^3$

$= \left(2\sqrt{3}-\frac{2}{3}\pi\right)a^3$

4 **(a)** The complex number z is given by $z = re^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta \le \frac{1}{2}\pi$.

(i) Given that $w = (\sqrt{3} - \mathrm{i})\,z$, find $|w|$ in terms of r and arg w in terms of θ. [2]

(ii) For a fixed value of r, draw on the same Argand diagram the the loci of z and w as θ varies. [2]

(iii) If $r = 1.5$ units, calculate (in terms of π) the length of the locus of w for $\mathrm{Im}(w) \ge 0$ as θ varies. [2]

(b) Sketch on a single Argand diagram the set of points representing all complex numbers v satisfying both of the following inequalities:

$$|v - 5 - 8\mathrm{i}| \le 5 \quad \text{and} \quad |v - 12 - 8\mathrm{i}| \ge |v - 12 - 10\mathrm{i}|.$$

Hence find (in radians) the least value of $\arg(v - 5 + 3\mathrm{i})$. [6]

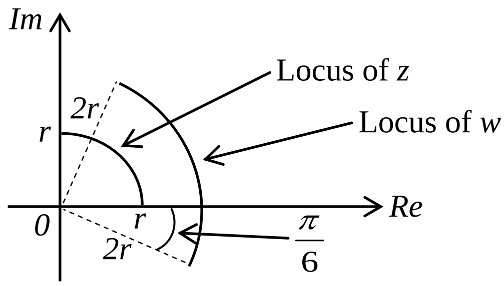
Suggested Solution

(a)

(i) $|w| = |\sqrt{3} - i||z| = \sqrt{(\sqrt{3})^2 + 1^2}\,r = 2r$

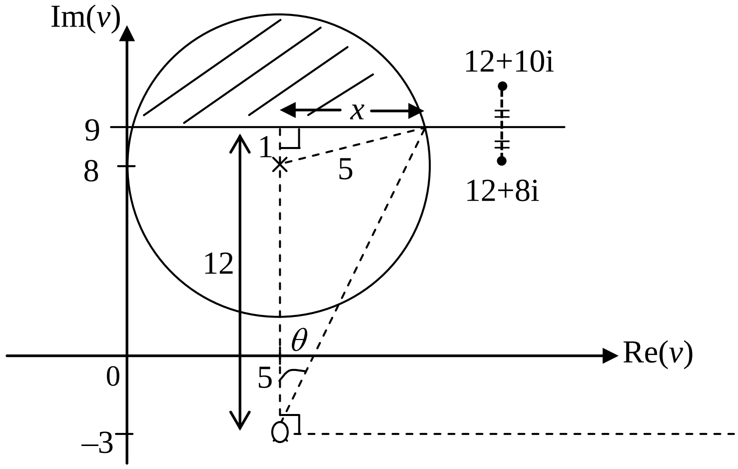
$\arg(w) = \arg\left(\sqrt{3} - i\right) + \arg(z) = \theta - \dfrac{\pi}{6}$

(ii)



(iii) Length of the locus of $w = \dfrac{1}{6}(2)(\pi)(3) = \pi$ units

(b)



From the diagram, $x^2 + 1^2 = 5^2 \quad \Rightarrow \quad x = \sqrt{24}$ [M1]

$$\tan\theta = \frac{\sqrt{24}}{12} \quad \Rightarrow \quad \theta = 0.3876 \text{ rad}$$ [M1]

Therefore, the least value of $\arg\left(v - 5 + 3\mathrm{i}\right) = \frac{\pi}{2} - 0.3876 = 1.18 \text{ rad}$ [A1]

Section B: Statistics [60 marks]

5 A company with eight hundred employees wishes to find out how much time its employees take to travel to work. It is given that the employees go to work either by car or by bus/train and that each of them takes the same form of transport to work every day.

The following table shows the number of employees going to work by car and the number of employees going to work by bus/train.

	Car	Bus/Train
Men	165	260
Women	82	293

The company wants to take a random sample of 180 employees.

(i) Explain what is meant, in this context, by the term “a random sample”. [1]

(ii) Describe how a random stratified sample can be obtained. [2]

(iii) Give a reason why quota sampling is not as suitable in this context compared to stratified sampling. [1]

Suggested Solution

(i) It means that every employee has an equal chance of being selected and that the selection of each employee is independent of one another.

(ii) Randomly assign numbers to each employee in each stratum:
Men taking car from 1 to 165;
Men taking bus/train from 1 to 260;
Women taking car from 1 to 82;
Women taking bus/train from 1 to 293.
Randomly select the required numbers for each stratum (as shown below) to form a sample of 180 employees.

(iii) Quota sampling does not produce a random sample, therefore it is not suitable in this context as compared to stratified sampling which produces a random sample as required by the company.

6 Salt is packed in bags to be sold. The manufacturer claims each bag contains at least 300 g of salt. To test this claim, a random sample of 15 bags of salt is examined and the mass, x g, of the contents of each bag is determined. It is found that the sample has a mean of 299.1 g and variance of 3.864 g^2.

(i) Test at the 10% significance level whether the manufacturer's claim is valid. [5]

(ii) State an assumption necessary for the test in **(i)** to be valid. [1]

Suggested Solution

(i) $\bar{x} = 299.1$ and unbiased estimate for the population variance is $s^2 = \frac{15}{14}(3.864) = 4.14$

Let μ denote the mean mass of salt in a randomly chosen bag.

To test H_o: $\mu = 300$

Against H_1: $\mu < 300$

Using 1 tailed test at 10% level of significance

Under H_o, test statistic $T = \dfrac{\bar{X} - 300}{\frac{S}{\sqrt{15}}} \sim t(14)$

Using GC, p – value = 0.0544.

Since p – value < 0.10, we reject H_o and conclude that there is sufficient evidence to that the mean mass of salt in a bag is less than 300 g, thus the manufacturer's claim is not valid at 10% level of significance.

(ii) The mass of salt in a randomly chosen bag is assumed to be following a normal distribution.

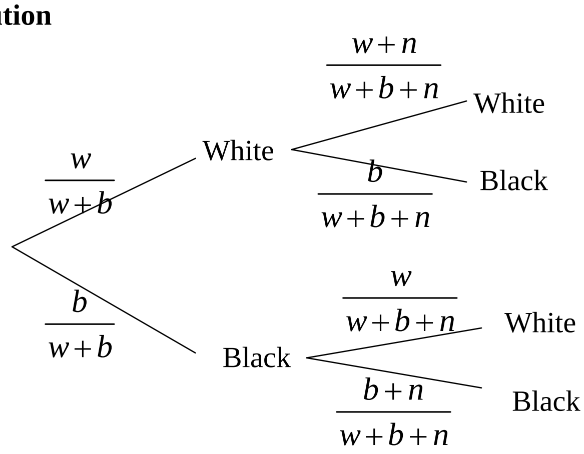
7 A bag contains w white balls and b black balls. One ball is selected at random from the bag, its colour noted and it is then returned to the bag along with n additional balls of the same colour. A second ball is then randomly selected from the bag.

(i) Construct a probability tree showing this information. [2]

(ii) Show that the probability that the second ball selected is black is independent of n. [2]

It is now known that the second ball drawn is black. Show that the probability that the first ball drawn is white is $\dfrac{w}{w+b+n}$. [2]

Suggested Solution



(ii) P(second ball is black)

$$= \frac{wb}{(w+b)(w+b+n)} + \frac{b(b+n)}{(w+b)(w+b+n)}$$

$$= \frac{b(w+b+n)}{(w+b)(w+b+n)}$$

$$= \frac{b}{w+b} \text{ is independent of } n$$

Required probability

$$= \frac{\text{P(first ball is white and second ball is black)}}{\text{P(second ball is black)}}$$

$$= \frac{\frac{wb}{(w+b)(w+b+n)}}{\frac{b}{w+b}} = \frac{w}{w+b+n} \text{ (shown)}$$

8 The table below shows Singapore's GDP per capita over the years from 1965.

Year, x	GDP per capita (in thousands), y
1965	1.580
1970	2.832
1975	6.607
1980	10.714
1985	14.921
1990	23.139
1995	35.346
2000	41.018
2005	49.715
2010	63.498

Source: Department of Statistics Singapore

(a) Using the data available,

(i) draw the scatter diagram, labelling the axes clearly, [2]

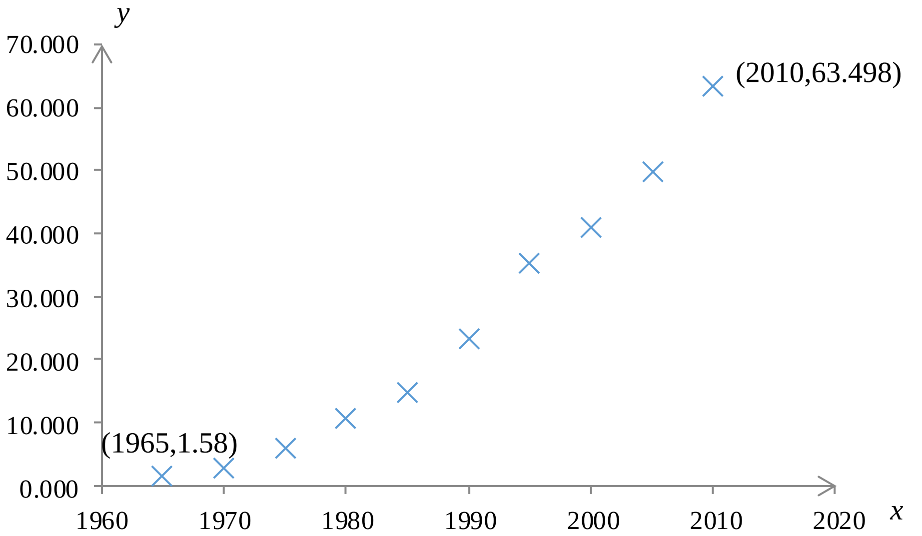
(ii) find the least square regression line of y on x, [1]

(iii) estimate the GDP per capita in the year 2015, correct to the nearest whole number. Comment on the reliability of the value obtained. [2]

(b) It is suggested that the data from 1965 to 2010 can be modelled by $y = a + b(x - 1965)^2$ instead of a linear model. Find the value of the product moment correlation coefficient for each of the proposed models and determine which is the better model. [2]

Suggested Solution

(i)



(ii) The regression line of y on x is $y = -2720.491909 + 1.381348x \Rightarrow y = -2720 + 1.38x$

(iii). The GDP per capita in 2015 is \$62 924. [accept variations]
Since this is an extrapolated value and the linear model may not hold beyond the data range, the value calculated is not reliable.

(b) Using a linear model $y = a + bx$, the product moment correlation coefficient, $r = 0.974$.

Using the quadratic model $y = a + b(x - 1965)^2$, the product moment correlation coefficient, $r = 0.996$.

Since the $|r|$ value for the quadratic model is closer to 1, this is a better model.

9 In a factory manufacturing calculators, it is found that 1.5% of the calculators produced are defective.

In a random sample of 90 calculators, find the probability that

(i) there are exactly 2 defective calculators, [1]

(ii) the 90th calculator is the second defective calculator given that there are exactly 2 defective calculators in that sample. [3]

The calculators are packed in boxes of 90.

(iii) Using a suitable approximation, find the probability that not more than 1 box of calculators, out of 60 boxes, contain more than 2 defective calculators. [5]

Suggested Solution

(i) Let X be the r.v. "number of defective calculators in a sample of 90".

$X \sim \mathrm{B}(90, 0.015)$

Using GC, $\mathrm{P}(X = 2) = 0.23833$

≈ 0.238 (to 3s.f.)

(ii) Let Y be the r.v. "number of defective calculators in a sample of 89".

$Y \sim \mathrm{B}(89, 0.015)$

Using GC,

required probability

$$= \frac{\mathrm{P}\left(90^{\text{th}} \text{ calculator is defective and } X = 2\right)}{\mathrm{P}(X = 2)}$$

$$= \frac{\mathrm{P}(Y = 1) \times \mathrm{P}\left(90^{\text{th}} \text{ calculator is defective}\right)}{\mathrm{P}(X = 2)}$$

$$= \frac{\mathrm{P}(Y = 1) \times 0.015}{\mathrm{P}(X = 2)}$$

$$= \frac{0.35308 \times 0.015}{0.23833} = 0.0222 \text{ (3s.f.)}$$

Alternatively,

Required probability

$$= \frac{\mathrm{P}\left(90^{\text{th}} \text{ calculator is defective and } X = 2\right)}{\mathrm{P}(X = 2)}$$

$$= \frac{{}^{89}C_1 \times (1-0.015)^{88} \times (0.015)^2}{0.23833} = 0.0222 \text{ (3s.f.)}$$

(iii) Let W be the r.v. "number of boxes, out of 60 boxes, with more than 2 defective calculators".

$W \sim \mathrm{B}(60, \mathrm{P}(X > 2)) \;\Rightarrow\; W \sim \mathrm{B}(60, 0.15338)$

Since $n = 60$ is large, $np = 9.2028 > 5$ and $nq = 50.7972 > 5$,

$W \sim \mathrm{N}(9.2028, 7.7913)$ approximately.

Using GC, $\mathrm{P}(W \le 1) \xrightarrow{c.c} \mathrm{P}(W < 1.5) = 0.0028938 \approx 0.00289$ (to 3s.f.)

10 Seven men and seven women, including Sally and Andy, participated in a speed-dating session at a community centre.

All participants are to sit in a way such that no two persons of the same gender sit next to each other.

How many ways can the participants be arranged if

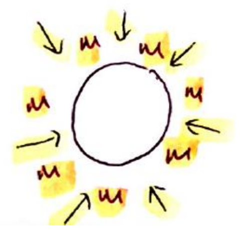
(i) they are seated at a round table of 14 seats? [2]

(ii) they are seated at 2 similar round tables of 6 seats each without Sally and Andy? [3]

(iii) they are seated on both sides of a rectangular table with 7 seats on each side, such that Sally and Andy sit next to each other on the same side? [3]

Suggested Solution

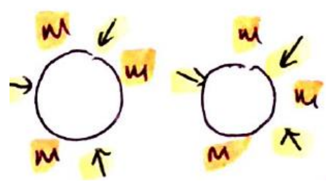
(i) No of ways $= (7-1)!7! \;= 3628800$



[arrange 7 men in a circle= $(7-1)!$ or $\frac{7!}{7}$]

[slot in the 7 women between the men $= 7!$]

(ii) No of ways= $\dfrac{{}^6C_3 \times \left[(3-1)!\right]^2 \times 6!}{2}$ = 28800

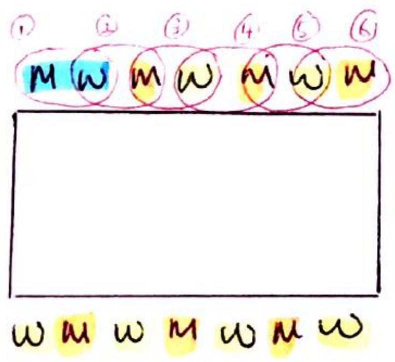


[out of 6 men, choose 3 to place at one table= ${}^6C_3$; note that the tables are similar, and hence there's double counting and thus need to divide by 2]

[arrange 3 men in a circle $= (3-1)!$; note that this value is squared]

[arrange 6 women in the remaining seats at both tables = 6!]

(iii) No of ways $= \dfrac{{}^6C_1 \times 2 \times 2}{2} \times 6! \times 6!$ =6220800



[on one side, choose a slot for Sally and Andy = ${}^6C_1$]
[there are two possible sides at table = 2]
[Sally and Andy can swop places = 2; when this happens all other couple will swop places as well because "no two persons of same gender are to sit beside one another" hence, there's double counting, thus need to dived by 2]
[there are 6 possible ways left to slot the rest of 6 women=6!]
[there are 6 possible ways then to slot the rest of 6 men=6!]

11 **(a)** A farm in the west of Singapore grows turnips for sale to the local market.

(i) 5 turnips are randomly chosen. Find the probability that exactly one turnip weighs less than the lower quartile weight and exactly two turnips weigh more than the median weight. [2]

(ii) The mass of a randomly chosen turnip has mean 40 g and standard deviation of 3 g. If the probability that the mean mass of a large sample of n turnips is greater than 39.6 g exceeds 0.95, find the least value of n. [3]

(b) A random variable X has the distribution $X \sim \mathrm{N}(40,\ 3^2)$. The random variable Y is related to X by the formula $Y = aX - \dfrac{1}{b}$, where a and b are constants and $a > 0$. Given that $\mathrm{P}(Y < 85) = \mathrm{P}(Y > 155) = 0.075$, find the values of $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$, and hence find the values of a and b. [5]

Suggested Solution

(a) (i) $\mathrm{P}(T < q_L).\mathrm{P}(T > m)^{\mathbf{2}}.[\mathrm{P}(q_L \le T \le m)]^2.\dfrac{5!}{2!2!}$

$= \left(\dfrac{1}{4}\right)\left(\dfrac{1}{2}\right)^2\left(\dfrac{1}{4}\right)^2 \dfrac{5!}{2!2!}$

= 0.0390625(30)

= 0.1171875

(ii)

Let T denotes the mass of a randomly chosen turnip.

Since n is large, $\overline{T} \sim N(40, \frac{3^2}{n})$ approximately by Central Limit Theorem.

$\mathrm{P}(\overline{T} > 39.6) > 0.95$

$1 - \mathrm{P}(\overline{T} \leq 39.6) > 0.95$

$\mathrm{P}(\overline{T} \leq 39.6) < 0.05$

Let $Z = \dfrac{\overline{T} - 40}{\frac{3}{\sqrt{n}}} \sim \mathrm{N}(0, 1)$

$$\mathrm{P}\left(Z \leq \frac{39.6 - 40}{\frac{3}{\sqrt{n}}} \right) < 0.05$$

$\dfrac{-2\sqrt{n}}{15} < -1.64485$

$\sqrt{n} > 12.336375$
$n > 152.1861$

The least value of n is 153.

(b)

$\mathrm{P}(Y < 85) = 0.075$

Since Y is a linear function of X, X is normal implies that Y is normal.

Since the normal distribution is symmetrical about the mean and
$\mathrm{P}(Y < 85) = \mathrm{P}(Y > 155) = 0.075$, $\mathrm{E}(Y) = \dfrac{85 + 155}{2} = 120$

$\mathrm{P}(Z < \dfrac{85 - 120}{\sigma}) = 0.075$

Using GC, $\mathrm{P}(Z < c) = 0.075 \Rightarrow c = -1.439531471$

$\dfrac{85 - 120}{\sigma} = -1.439531471$

$\sigma = 24.31346636$ and $\text{Var}(Y) = \sigma^2 = 591.1446464$

$\text{Var}(Y) = a^2\text{Var}(X) + \text{Var}(\frac{1}{b}),$ where $\text{Var}(\frac{1}{b}) = 0$

$\text{Var}(Y) = 9a^2$

$a = \pm 8.104488786 = 8.10$ (reject negative since $a > 0$)

$\text{E}(Y) = a\text{E}(X) - \frac{1}{b}$

$120 = 40a - \frac{1}{b}$

$\frac{1}{b} = 40(8.10449) - 120 = 204.17955$

$b = 0.00490$ (to 3 s.f.)

12 **(a)** A teacher discovered that the probability that a randomly chosen student is late two days in a month is three times the probability that the student is late four days in a month. If the number of days a student is late in a month follows a Poisson distribution, find the non-zero variance of the Poisson distribution. [3]

(b) Records indicate that a certain hospital delivers an average of 3650 babies each year. Each day, there are 3 shifts of equal duration in the hospital. It is assumed that the number of deliveries in a day can be modelled by a Poisson distribution. Taking a year to consist of 365 days,

(i) show that the mean number of deliveries per shift is $\frac{10}{3}$ and find the most likely number of deliveries per shift, [3]

(ii) find the expected number of shifts with at least 5 deliveries in a week, [3]

(iii) explain why a Poisson distribution may not be a suitable model for the number of deliveries in a day for a hospital with a significant number of planned deliveries (e.g. cesarean section). [1]

Suggested Solution

(a) Let X denote the random variable representing the number of times a student is late in a month.

$\therefore\ X \sim \text{Po}(\lambda)$ where $\lambda = \text{Var}(X)$

$\text{P}(X = 2) = 3\ \text{P}(X = 4)$

$$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{3e^{-\lambda}\lambda^4}{4!}$$

$$\Rightarrow 1 = \frac{\lambda^2}{4}$$

$\Rightarrow \lambda = 2$ (since $\lambda > 0$)

(b) (i) Mean number of deliveries per shift each day

$$= \frac{3650}{365\times 3} = \frac{10}{3}$$

Let X denote the random variable representing the number of deliveries in a shift.

Then, $X \sim \text{Po}\left(\frac{10}{3}\right)$

From the GC,

x	$\text{P}(X = x)$
2	0.19819
3	0.22021
4	0.18351

$\therefore$ most likely number of deliveries per shift is 3. [M1, A1]

(ii) $\text{P}(X \geq 5)$
$= 1 - \text{P}(Y \leq 4)$
$= 1 - 0.75649$
$= 0.24351$

Let Y denote the random variable representing the number of shifts with at least 5 deliveries. i.e. $Y \sim \text{B}(21, 0.24351)$
$\therefore$ the expected number of shifts with more than 5 deliveries in a week
$= 0.24351 \times 21$
$= 5.11$

(iii) Planned deliveries (e.g. cesarean section) are not natural child births and are less likely to be scheduled at night. Hence, a Poisson distribution may not be a suitable model for the number of deliveries in a day as the average number of deliveries per shift may not be constant.