RAFFLES INSTITUTION
H2 Mathematics (9758)
2017 Year 6

2017 H2 Math 9758 Preliminary Examination Paper 1 : Suggested Solutions

Qn 1 [4]

Let \$$x$ and \$$y$ be the price of each small and each large Pikachi plushy respectively.

Retailer A :

$$30x + 50y = 1375 \quad \ldots (1)$$

Retailer B :

$$kx + 2ky = 2704$$ ÷ by k

$$x + 2y - 2704\left(\frac{1}{k}\right) = 0 \quad \ldots (2)$$

Retailer C :

$$2kx + ky = 2522$$ ÷ by k

$$2x + y - 2522\left(\frac{1}{k}\right) = 0 \quad \ldots (3)$$

From GC : $x = 15$, $y = 18.5$, $\frac{1}{k} = \frac{1}{52}$

Hence, $k = 52$, each small Pikachi plushy costs \$15, and each large Pikachi plushy costs \$18.50.

Comments: Most students were able to form the three equations. Only a handful realized the need to linearise the equations before using GC to solve for the unknowns.

Qn 2 [4]

Let $A = \pi r\sqrt{r^2 + h^2}$. square both sides

$$\Rightarrow A^2 = \pi^2 r^2\left(r^2 + h^2\right)$$

Differentiate w.r.t. t :

$$r^2\left(2r\frac{dr}{dt} + 2h\frac{dh}{dt}\right) + \left(r^2 + h^2\right)\left(2r\frac{dr}{dt}\right) = 0$$

(Note: $\frac{dA}{dt} = 0$ since A is a constant)

Since $r \neq 0$, $\left(2r^2 + h^2\right)\frac{dr}{dt} + hr\frac{dh}{dt} = 0$

$$\Rightarrow \left(\frac{dh}{dt}\right) \div \left(\frac{dr}{dt}\right) = \frac{2r^2 + h^2}{-hr}$$

When $r = \sqrt{2}$, $\frac{dh}{dt} = -10\frac{dr}{dt} \Rightarrow \frac{4 + h^2}{-\sqrt{2}h} = -10$

$$\Rightarrow h^2 - 10\sqrt{2}h + 4 = 0$$

Solving: $h = 13.9$ (3sf) or $h = 0.289$ (3sf)

Since $h > r$, the height of the cone required is 13.9 cm (to 3 sf).

Comments: Students must remember that r and h are variables and not constants. When performing implicit differentiation on the variables with respect to r , h or t , you must have your $\frac{dh}{dr}$ or $\frac{dr}{dt}$ etc.

Do remember to substitute the given conditions **after** differentiation and not ***before*** differentiation!

Qn 3

(a) [4]

$$\int \frac{x+2}{\sqrt{1-8x-4x^2}}\,dx$$

$$= -\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{1}{\sqrt{1-8x-4x^2}}\,dx$$

$$= -\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{1}{\sqrt{5-4(x+1)^2}}\,dx$$ (complete the square)

$$= -\frac{1}{4}\sqrt{1-8x-4x^2} + \frac{1}{2}\sin^{-1}\frac{2\sqrt{5}(x+1)}{5} + C$$

Comments: This question is not well done. Most split the integrand to

$$\int \frac{x}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{2}{\sqrt{1-8x-4x^2}}\,dx$$

and have difficulty integrating the first term.
There are quite a few who wrote

$$-\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx = -\frac{1}{8}\frac{\left(1-8x-4x^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

(b) [4]

$x = 2\sec\theta \Rightarrow \frac{dx}{d\theta} = 2\sec\theta\tan\theta$

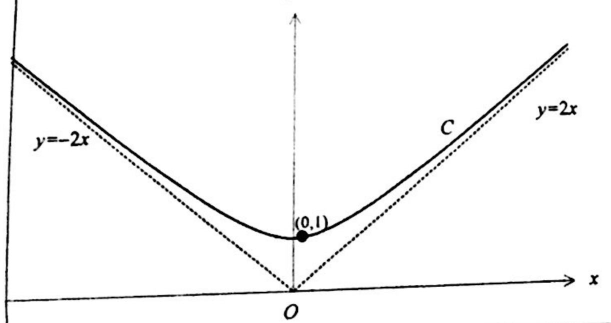
When $x = 2$, $\sec\theta = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

When $x = 4$, $\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

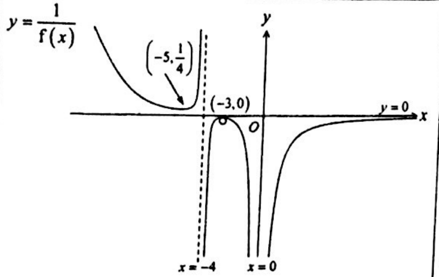
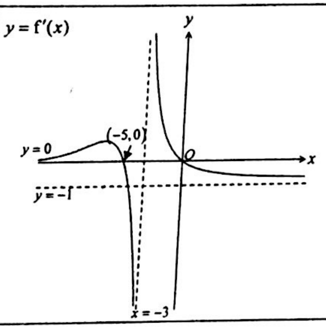
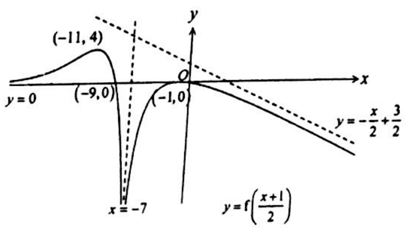
$$\int_2^4 \frac{1}{x}\sqrt{\left(x^2-4\right)}\,dx$$

Comments: There is a significant number of students who do not know how to integrate $\tan^2\theta$ and $\sec^2\theta$. Make sure you know how to integrate all the trigonometric function ($\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$) and $(\text{trigonometric})^2$ function, such as $\sin^2 x, \cos^2 x, \tan^2 x, \sec^2 x, \operatorname{cosec}^2 x, \cot^2 x$, and be familiar with the formulae/identities given in MF26.

	$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta$ $= \int_0^{\frac{\pi}{3}} \frac{2\tan\theta}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta$ since $\sqrt{4\sec^2\theta-4} = 2\sqrt{\tan^2\theta} = 2\tan\theta$ for $0 \le \theta \le \frac{\pi}{3}$ $= \int_0^{\frac{\pi}{3}} 2\tan^2\theta\,d\theta$ $= 2\int_0^{\frac{\pi}{3}} \sec^2\theta - 1\,d\theta$ $= 2\left[\tan\theta - \theta\right]_0^{\pi/3}$ $= 2\left[\sqrt{3} - \frac{\pi}{3}\right]$	The other common mistake is not changing the limits of the integration when substituting x by $2\sec\theta$.

Qn 4		Comments
(i) [4]	$y = \frac{a}{(x+b)^2} + cx$ C has a vertical asymptote $x=-1 \Rightarrow b=1$ C passes through $(0,1) \Rightarrow a=1$ $\frac{dy}{dx} = -\frac{2}{(x+1)^3} + c$ At $(0,1)$, $\frac{dy}{dx} = 0 \Rightarrow c = \frac{2}{1^3} = 2$	When finding the derivative of $\frac{a}{(x+b)^2}$ with respect to x, there is no need to use the quotient rule. If the unknown constants bother you, ask yourself how you would proceed to find its derivative if you assume some real values for the unknown constants. Eg $\frac{1}{(x+1)^2}$ If you can identify the values of some of the unknowns immediately, substituting these values into the original expression will help to simplify your calculations.
(ii) [3]		Whenever you are sketching a graph, you should always remember SIA (shape, intercepts, asymptotes). Question also states that (0, 1) is a minimum point, so if it is not featured in the graph, something is wrong! Your graph should also be symmetrical about the y-axis.

(iii) [2]	$f(\lvert x\rvert) - 4 > 0 \Leftrightarrow f(\lvert x\rvert) > 4$ The line $y=4$ cuts the graph of $y = f(\lvert x\rvert)$ at $x = \pm 1.94$ (3sf). $\therefore f(\lvert x\rvert) - 4 > 0 \Leftrightarrow x < -1.94$ or $x > 1.94$	Question did not ask for exact answers, so it is not necessary to solve for the intersection points algebraically. You just need to plot a graph and find its intersection with the x-axis using a GC. Your final answer should be symmetrical about the y axis, and remember to give the final non exact answers to 3 sf.

Qn 5		Comments
(a) [3]	$y=\frac{1}{f(x)}$ 	**General** Co-ordinates are required for axial intercept(s) and turning point(s). We want to see $(-3,0)$ marked in part (a), and so on. Equations of asymptotes must be labelled. Many students left out "$y=0$" and/or "$x=0$". Curve sketched near asymptote must exhibit the characteristic of approaching nearer and nearer to it. Use ruler to draw (dotted) straight lines to represent asymptotes! **For (a)** : An empty circle ought to be marked at the point $(-3,0)$ to indicate the exclusion of the point. **For (c)** : We can't leave the equation of the oblique asymptote as $y=-\frac{x+1}{2}+2$. Simplify it. Many students had the wrong translation unit and/or scaling factor. There are 2 options for your consideration : (1) $f(x)\to f\left(\frac{x}{2}\right)\to f\left(\frac{x+1}{2}\right)$ (2) $f(x)\to f\left(x+\frac{1}{2}\right)\to f\left(\frac{x}{2}+\frac{1}{2}\right)$
(b) [3]	$y=f'(x)$ 	
(c) [3]		

Qn 6		Comments
(i) [5]	$y=\ln(1+\sin 2x)$ so $e^y=1+\sin 2x$ Differentiating with respect to x: $e^y\frac{dy}{dx}=2\cos 2x$ $e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=-4\sin 2x$ (shown) $e^y\frac{d^3y}{dx^3}+e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+2e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+e^y\left(\frac{dy}{dx}\right)^3=-8\cos 2x$ $e^y\frac{d^3y}{dx^3}+3e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+e^y\left(\frac{dy}{dx}\right)^3=-8\cos 2x$ When $x=0$, $e^y=1\Rightarrow y=0$, $\frac{dy}{dx}=2$, $\frac{d^2y}{dx^2}=-4$, $\frac{d^3y}{dx^3}=8$. By Maclaurin's Theorem, $y=2x-4\left(\frac{x^2}{2!}\right)+8\left(\frac{x^3}{3!}\right)+\cdots$ $=2x-2x^2+\frac{4}{3}x^3+\cdots$	Since we need to derive the 2[nd] order DE which involves e^y, we should express $y=\ln(1+\sin 2x)$ as $e^y=1+\sin 2x$ and apply implicit differentiation. Direct differentiation can be complicated at times. Note that $\frac{d}{dx}\left(\frac{dy}{dx}\right)^2=2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$ An alternative solution involves applying standard Maclaurin expansion although this is not the intended method.
(ii) [5]	$ax(1+bx)^n$ $=ax\left[1+n(bx)+\frac{n(n-1)}{2!}(bx)^2+\frac{n(n-1)(n-2)}{3!}(bx)^3+\ldots\right]$ $=ax+nabx^2+\frac{n(n-1)}{2}ab^2x^3+\ldots$ By comparing coefficients, $a=2$ $nab=-2\Rightarrow nb=-1$	A common mistake involves omitting the term, ax, in the expansion of $ax(1+bx)^n$. Also, since we do not know whether n is a positive integer or not, we should not use $\binom{n}{r}$ or $n!$.

	$\frac{n(n-1)}{2}ab^2=\frac{4}{3}$ $\Rightarrow n^2b^2-nb^2=\frac{4}{3}$ $\Rightarrow (-1)^2-(-1)b=\frac{4}{3}$ $\Rightarrow b=\frac{1}{3}$ and $n=-3$ x^4 term in the expansion of $2x\left(1+\frac{1}{3}x\right)^{-3}$ $=2x\left[\frac{-3(-4)(-5)}{3!}\left(\frac{1}{3}x\right)^3\right]=-\frac{20}{27}x^4$ $\therefore$ coefficient of $x^4=-\frac{20}{27}$	

Qn 7		Comments
(i) [1]	Let β be the angle of elevation of the bottom of the screen from eye-level. $\tan(\alpha+\beta)=\frac{a+b}{x}\Rightarrow \alpha+\beta=\tan^{-1}\frac{a+b}{x}$ $\tan(\beta)=\frac{b}{x}\Rightarrow \beta=\tan^{-1}\frac{b}{x}$ $\alpha=(\alpha+\beta)-\beta=\tan^{-1}\frac{a+b}{x}-\tan^{-1}\frac{b}{x}$	Students should define the angles properly. They can also draw a diagram to indicate the angles α and β.
(ii) [4]	$\frac{d\alpha}{dx}=\frac{1}{1+\left(\frac{a+b}{x}\right)^2}\left(-\frac{a+b}{x^2}\right)-\frac{1}{1+\left(\frac{b}{x}\right)^2}\left(-\frac{b}{x^2}\right)$ $=-\frac{a+b}{x^2+(a+b)^2}+\frac{b}{x^2+b^2}$ For maximum α: $\frac{d\alpha}{dx}=-\frac{(a+b)}{x^2+(a+b)^2}+\frac{b}{x^2+b^2}=0$ $\frac{(a+b)}{x^2+(a+b)^2}=\frac{b}{x^2+b^2}$ (Shown) $(a+b)(x^2+b^2)=b[x^2+(a+b)^2]$ $(a+b)x^2+(a+b)b^2=bx^2+b(a+b)^2$ $ax^2=b(a+b)[a+b-b]=ab(a+b)$ $x=\sqrt{b(a+b)}$ or $-\sqrt{b(a+b)}$ (NA since $x>0$)	Students should note the application of chain rule in the showing of the differentiation in the first line.

(iii) [5]	Let the screen to be positioned y metres above the eye level. $\alpha=\tan^{-1}\frac{a+y}{c}-\tan^{-1}\frac{y}{c}$ $\frac{d\alpha}{dy}=\frac{1}{1+\left(\frac{a+y}{c}\right)^2}\left(\frac{1}{c}\right)-\frac{1}{1+\left(\frac{y}{c}\right)^2}\left(\frac{1}{c}\right)$ $=\frac{c}{c^2+(a+y)^2}-\frac{c}{c^2+y^2}$ $=\frac{c\left[c^2+y^2-c^2-(a+y)^2\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$ $=\frac{c\left[y+(a+y)\right]\left[y-(a+y)\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$ $=\frac{-ac[a+2y]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$ $=\frac{-2ac\left[y+\frac{a}{2}\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$ For maximum α: $\frac{d\alpha}{dy}=\frac{-ac[a+2y]}{\left[c^2+(a+y)^2\right](c^2+y^2)}=0$ $\Rightarrow y=-\frac{a}{2}$ (since $a\neq 0$ and $c\neq 0$)	Students should note that it is eye level and not ground level given in the question, for those who concluded that it is impossible to get y being negative.

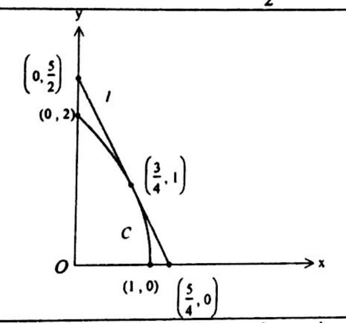
y	$\left(-\frac{a}{2}\right)^-$	$\left(-\frac{a}{2}\right)$	$\left(-\frac{a}{2}\right)^+$
$\left[y+\frac{a}{2}\right]$	<0	0	>0
$-2ac\left[y+\frac{a}{2}\right]$, where $-2ac<0$	>0	0	<0
$\frac{d\alpha}{dy}$	>0	0	<0

Therefore $y=-\frac{a}{2}$ gives the maximum viewing angle α.

Interpretation of the answer:
In order to maximise the viewing angle α, the centre of the screen need to be placed at eye level regardless of the position of the sofa.

Comments: Students should show how they deduced the signs of the derivative for $y=\left(-\frac{a}{2}\right)^-$ and $y=\left(-\frac{a}{2}\right)^+$. Some were also successful in using the second derivative test, although it is more demanding to obtain.

Qn 8		Comments
(i) [2]	Using $\sin^2 t+\cos^2 t=1$, a cartesian equation of C is	Important to remember the Trigo identities. Answers such as $y=2\cos\left(\sin^{-1}\sqrt{x}\right)$

	$x+\left(\frac{y}{2}\right)^2=1$ $\Rightarrow y^2=4-4x,\ 0\le x\le 1,\ 0\le y\le 2$ or $\Rightarrow y=2\sqrt{1-x},\ 0\le x\le 1$	are **not** accepted as they are not simplified. Essential to state $0\le x\le 1$ and/or $0\le y\le 2$ as C is defined for $0\le t\le \frac{\pi}{2}$.
(ii) **[3]**	Differentiate with respect to x: $1+\frac{y}{2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{2}{y}$ When $t=\frac{\pi}{3}$, $x=\sin^2\left(\frac{\pi}{3}\right)=\frac{3}{4}$, $y=2\cos\left(\frac{\pi}{3}\right)=1$, $\frac{dy}{dx}=-2$ Hence, an equation of l is $y-1=-2\left(x-\frac{3}{4}\right)$ $y=-2x+\frac{5}{2}$	$\frac{dy}{dx}$ can also be found by parametric or explicit differentiation.
(iii) **[3]**		Sketch C for $0\le x\le 1$ only. State the coordinates of the point of intersection and axial intercepts. Sketch should illustrate that line l is a tangent to curve C at $\left(\frac{3}{4},1\right)$.
(iv) **[3]**	Volume of revolution of R rotated about the x-axis $=\pi\int_0^{\frac{3}{4}}\left(-2x+\frac{5}{2}\right)^2 dx-\pi\int_0^{\frac{3}{4}}(4-4x)\,dx$ $=\pi\left[\frac{\left(-2x+\frac{5}{2}\right)^3}{3(-2)}\right]_0^{\frac{3}{4}}-\pi\left[4x-2x^2\right]_0^{\frac{3}{4}}$ $=-\frac{1}{6}\pi\left[1^3-\left(\frac{5}{2}\right)^3\right]-\pi\left[3-2\left(\frac{3}{4}\right)^2\right]$	As "exact value" is required, you are to **show clear working** instead of using the GC to obtain the values of the integrals. (You may, of course, use the GC to check your answer).

	$=\frac{39}{16}\pi-\frac{15}{8}\pi$ $=\frac{9}{16}\pi$ units^3 OR Use volume of cone $=\frac{1}{3}\pi r^2 h$, i.e. $\left[\frac{1}{3}\pi\left(\frac{5}{2}\right)^2\left(\frac{5}{4}\right)-\frac{1}{3}\pi(1)^2\left(\frac{1}{2}\right)\right]-\pi\int_0^{\frac{3}{4}}(4-4x)\,dx$	

Qn 9		Comments
(a) **(i)** **[5]**	Since $z=1+3i$ is a root and the polynomial has real coefficients, $z=1-3i$ is also a root to the polynomial. Hence a quadratic factor of the polynomial is $(z-(1+3i))(z-(1-3i))=$ $(z^2-z(1+3i+1-3i)+(1+3i)(1-3i))=(z^2-2z+10)$ $z^4+2z^3+az^2+bz+50$ $=(z^2-2z+10)(Az^2+Bz+C)$ for some constants A, B and C. By comparing coefficient of z^4 and z^3, $A=1$ and $B-2A=2\Rightarrow B=4$ By comparing the constant term, $C=5$ Hence $z^4+2z^3+az^2+bz+50=(z^2-2z+10)(z^2+4z+5)$ Comparing coefficient of z^2 and z, we have $a=-8+10+5=7$ and $b=40-10=30$ (shown). Solving $z^2+4z+5=0$, $z=\frac{-4\pm\sqrt{4^2-4(5)}}{2}=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm\sqrt{4}\sqrt{-1}}{2}=-2\pm i$. Hence the other roots are $z=1-3i$, $z=-2+i$ and $z=-2-i$. Alternative Solution (more tedious): Since $1+3i$ is a root, $(1+3i)^4+2(1+3i)^3+a(1+3i)^2+b(1+3i)+50=0$ --(1) $(1+3i)^2=1^2+2(3i)+(3i)^2=(1-9)+6i=-8+6i$ $(1+3i)^3=(1+3i)(-8+6i)=(-8-18)+i(6-24)=-26-18i$	Note the following instruction is given at the start of the question - **"Do not use a calculator in answering this question"**. Obviously you can use GC to check your answer, but you are required to show clear working.

$(1+3i)^4 = (-8+6i)^2 = 64-96i-36 = 28-96i$

Applying above results on (1),

$(28-96i)+2(-26-18i)+a(-8+6i)+b(1+3i)+50=0$

$(26-8a+b)+(-132+6a+3b)i=0$

Comparing real and imaginary parts,

$26-8a+b=0$ and $-132+6a+3b=0$

equivalent to $-44+2a+b=0$

Solving, $-44-26+10a=0 \Rightarrow a=7$ and $b=8(7)-26=30$

$\therefore a=7,\ b=30$ (shown)

Since $z=1+3i$ is a root and the polynomial has real coefficients, $z=1-3i$ is also a root to the polynomial.

$z^4+2z^3+7z^2+30z+50$

$=(z-(1+3i))(z-(1-3i))(z^2+Az+B)$

$=(z^2-2z+10)(z^2+Az+B)$

By comparing coefficients, we have $A=4$, $B=5$.

Solving $z^2+4z+5=0$, $z=\dfrac{-4\pm\sqrt{-4}}{2}=\dfrac{-4\pm\sqrt{4}\sqrt{-1}}{2}=-2\pm i$.

Hence the other roots are $z=1-3i$, $z=-2+i$ and $z=-2-i$.

(ii) [2]

Let $z=iw$, then we get $(iw)^4+2(iw)^3+7(iw)^2+30(iw)+50=0$

$\Rightarrow w^4-2iw^3-7w^2+30iw+50=0$.

$z=iw \Rightarrow w=-iz$.

Hence the roots are $w=-i-3$, $w=-i+3$, $w=2i+1$ and $w=2i-1$.

Comment: Note that $\dfrac{1}{i}=-i$

(b) [4]

$|p|=|p^*|=\dfrac{\left|\left(-\frac{1}{\sqrt{3}}+i\right)\right|^5}{|(1-i)|^4}=\dfrac{\left(\frac{2}{\sqrt{3}}\right)^5}{(\sqrt{2})^4}=\dfrac{32}{4}\left(\dfrac{1}{\sqrt{3}}\right)^5=\dfrac{8}{9\sqrt{3}}$ or $\dfrac{8\sqrt{3}}{27}$

$\arg(p)=-\arg(p^*)=-\left(5\arg\left(-\frac{1}{\sqrt{3}}+i\right)-4\arg(1-i)\right)+2\pi+2\pi$

$=-\left(5\left(\frac{2\pi}{3}\right)-4\left(-\frac{\pi}{4}\right)\right)+2\pi+2\pi$

$=-\dfrac{\pi}{3}$

$p=\dfrac{8}{9\sqrt{3}}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)=\dfrac{8}{9\sqrt{3}}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\dfrac{4}{9\sqrt{3}}-\dfrac{4}{9}i$ or $\dfrac{4\sqrt{3}}{27}-\dfrac{4}{9}i$

Comments: Note that this question requires you to consider $|p^*|$ and $\arg(p^*)$.

Remember to simplify your surds.

Note that $\arg(1-i)=-\dfrac{\pi}{4}$

Qn 10

(a)(i) [6]

$\dfrac{dx}{dt}=\dfrac{1}{10}x(1-x)$

$\Rightarrow \displaystyle\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx=\int\frac{1}{10}dt$

$\Rightarrow \ln\left|\dfrac{x}{1-x}\right|=\dfrac{1}{10}t+C$,

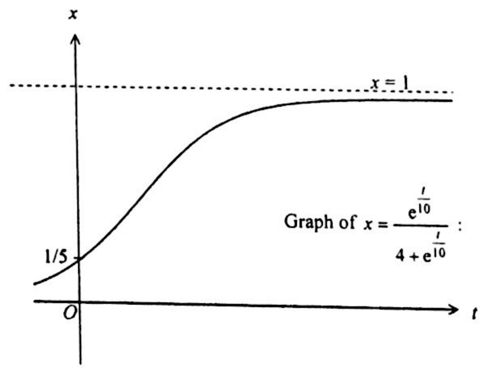
where C is an arbitrary constant

$\Rightarrow \dfrac{x}{1-x}=Ae^{\frac{t}{10}}$, where $A=\pm e^C$

$\Rightarrow x=\dfrac{Ae^{\frac{t}{10}}}{1+Ae^{\frac{t}{10}}}$

When $t=0$, $x=\dfrac{1}{5}$. That is, $\dfrac{1}{5}=\dfrac{A}{1+A}\Rightarrow A=\dfrac{1}{4}$.

Hence, $x=\dfrac{\frac{1}{4}e^{\frac{t}{10}}}{1+\frac{1}{4}e^{\frac{t}{10}}}=\dfrac{e^{\frac{t}{10}}}{4+e^{\frac{t}{10}}}=1-\dfrac{4}{4+e^{\frac{t}{10}}}$.



Comments: If $\ln\dfrac{x}{1-x}$ is given instead of $\ln\left|\dfrac{x}{1-x}\right|$, then $0<x<1$ should be stated to justify the removal of the modulus.

Final answer should be in simplified form, it should not contain a fraction within a fraction like $\dfrac{\frac{1}{4}e^{\frac{t}{10}}}{1+\frac{1}{4}e^{\frac{t}{10}}}$.

(ii) [2]

Since $\dfrac{dx}{dt}=\dfrac{1}{10}x(1-x)$ is a quadratic expression, destruction rate is at its maximum when $x=\dfrac{0+1}{2}=\dfrac{1}{2}$

Or when $\dfrac{d}{dx}\left(\dfrac{dx}{dt}\right)=\dfrac{1}{10}-\dfrac{1}{5}x=0$ i.e., $x=\dfrac{1}{2}$.

Comments: The "**destruction rate** is at its maximum" refers to maximum

	Therefore, $\dfrac{\frac{1}{2}}{1-\frac{1}{2}}=\dfrac{1}{4}e^{\frac{t}{10}}$. $\Rightarrow \quad e^{\frac{t}{10}}=4 \Rightarrow \quad t=10\ln 4$	$\dfrac{dx}{dt}$, not maximum x.
(iii) [1]	From $x=\dfrac{e^{\frac{t}{10}}}{4+e^{\frac{t}{10}}}$, $\quad x=0 \Rightarrow e^{\frac{t}{10}}=0$. Since $e^{\frac{t}{10}}>0$ for all real t, there is no value of t for $x=0$. **OR** Note that $\quad x=1-\dfrac{4}{4+e^{\frac{t}{10}}}$. Since $0<\dfrac{4}{4+e^{\frac{t}{10}}}<1$, we will have $0<x<1$. Hence, $x>0$ for all real values of t, and there is no value of t for $x=0$. **OR** As $x\to-\infty$, $e^{\frac{t}{10}}\to 0^+$, $x=1-\dfrac{4}{4+e^{\frac{t}{10}}}\to 0^+$. Hence, $x=0$ is a horizontal asymptote and there are no values of t giving $x=0$.	The question asks to explain why the model **cannot** be used to estimate (i.e. why we are unable to estimate using the model), it does not ask for why the model may not give a good estimate. So answers like "extrapolation is not reliable" or "the model is not valid for $t<0$" are not accepted.
(b) [3]	$\dfrac{dx}{dt}=\dfrac{1}{5\pi\left[1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2\right]}$ $x=\dfrac{1}{5\pi}\displaystyle\int \dfrac{1}{1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2}\,dt$ $=\dfrac{10}{5\pi}\displaystyle\int \dfrac{\frac{1}{10}}{1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2}\,dt$	

	$=\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{10}+\tan\dfrac{\pi}{10}\right)+C$ When $t=0$, $x=\dfrac{1}{5}$. Hence $\dfrac{1}{5}=\dfrac{2}{\pi}\tan^{-1}\left(\tan\dfrac{\pi}{10}\right)+C \Rightarrow C=0$ That is, $x=\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{10}+\tan\dfrac{\pi}{10}\right)$. From G.C., when $x=0$, $t=-3.25$ (3 s.f.) Hence, the forest have been burning for 3.25 hours when it is first noticed.	It is important to have "+ C" then show that $C=0$. Without this step, no mark can be awarded for the final answer.

Qn 11		Comments
(i) [2]	$\overrightarrow{OP}=\begin{pmatrix}20\\-4\\0\end{pmatrix}$, $\overrightarrow{OV}=\begin{pmatrix}0\\0\\2h\end{pmatrix}$ $\overrightarrow{PV}=\begin{pmatrix}-20\\4\\2h\end{pmatrix}=2\begin{pmatrix}-10\\2\\h\end{pmatrix}$ Vector equation of the line depicting the path of the light ray from P to V is $\mathbf{r}=\begin{pmatrix}20\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-10\\2\\h\end{pmatrix}$, $\lambda\in\mathbb{R}$	Note that the vector equation of a line is of the form: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$ It is not written as $l=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$ or equation of line $=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$
(ii) [3]	$\mathbf{r}.\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$ $\mathbf{r}=\begin{pmatrix}20\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-10\\2\\h\end{pmatrix}$, $\lambda\in\mathbb{R}$ $\begin{pmatrix}20-10\lambda\\-4+2\lambda\\\lambda h\end{pmatrix}.\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$ $20-10\lambda=\alpha\Rightarrow\lambda=\dfrac{20-\alpha}{10}$	"Does not exceed" means that the height is ≤ 35 units. Students who attempted this question by similar triangles must take note that α is a negative value.

	For shadow of the pyramid cast on the screen to not exceed the height of the screen, length of shadow, $\lambda h=\left(\frac{20-\alpha}{10}\right)h\le 35$ $\Rightarrow h\le\frac{350}{20-\alpha}$ since $\alpha<-4$ implies $20-\alpha>0$	
(iii) [3]	Given that h = 10 $\overrightarrow{OB}=\begin{pmatrix}4\\-4\\0\end{pmatrix}$, $\overrightarrow{OV}=\begin{pmatrix}0\\0\\20\end{pmatrix}\Rightarrow\overrightarrow{BV}=\begin{pmatrix}-4\\4\\20\end{pmatrix}$ Length of the shadow cast by edge VB $=\left\|\begin{pmatrix}-4\\4\\20\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}\right\|=\left\|\begin{pmatrix}0\\20\\-4\end{pmatrix}\right\|=\sqrt{416}=4\sqrt{26}$	Please note that the shadow cast by the edge VB on the screen is <u>not</u> $\left\|\overrightarrow{VB}\right\|$.
(iv) [4]	$\overrightarrow{OC}=\begin{pmatrix}4\\4\\0\end{pmatrix}$, $\overrightarrow{OV}=\begin{pmatrix}0\\0\\20\end{pmatrix}$, $\Rightarrow\overrightarrow{CV}=\begin{pmatrix}-4\\-4\\20\end{pmatrix}=-4\begin{pmatrix}1\\1\\-5\end{pmatrix}$ $\overrightarrow{BV}=\begin{pmatrix}-4\\4\\20\end{pmatrix}=4\begin{pmatrix}-1\\1\\5\end{pmatrix}$ ~~$\overrightarrow{BV}\times\overrightarrow{CV}$~~ A vector normal to plane VBC is $=\begin{pmatrix}1\\1\\-5\end{pmatrix}\times\begin{pmatrix}-1\\1\\5\end{pmatrix}=\begin{pmatrix}10\\0\\2\end{pmatrix}=2\begin{pmatrix}5\\0\\1\end{pmatrix}$ Vector equation of the plane VBC is $\mathbf{r}.\begin{pmatrix}5\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\20\end{pmatrix}.\begin{pmatrix}5\\0\\1\end{pmatrix}\Rightarrow\mathbf{r}.\begin{pmatrix}5\\0\\1\end{pmatrix}=20$ Angle of inclination made by the mirror with the ground	Students must be more careful when computing vectors. There is a lot of computation error for $\overrightarrow{CV}$ and $\overrightarrow{BV}$. This is just a direct application of angle between 2 planes with the normal of the ground (x-y plane) taken to be $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Note that this angle of inclination is **<u>not</u>** the same

	is $\cos^{-1}\left\|\frac{\begin{pmatrix}0\\0\\1\end{pmatrix}.\begin{pmatrix}5\\0\\1\end{pmatrix}}{\sqrt{1}\sqrt{25+1}}\right\|=\cos^{-1}\left\|\frac{1}{\sqrt{26}}\right\|=78.7^\circ$ (correct to 1 d.p.)	as the angle between $\overrightarrow{BV}$ and $\overrightarrow{BA}$ which is equal to 78.9°