2023 JC 1 H2 Further Mathematics (9649)

Promotion Examination Solution

1(a)	Consider $\left(\sqrt{2},2,0\right)$ and $\left(\sqrt{3},-\sqrt{6},0\right)\in\ W$. $\left(\sqrt{2},2,0\right)+\left(\sqrt{3},-\sqrt{6},0\right)=\left(\sqrt{2}+\sqrt{3},2-\sqrt{6},0\right)$ $2\left(\sqrt{2}+\sqrt{3}\right)^2-\left(2-\sqrt{6}\right)^2$ $=2\left(2+2\sqrt{6}+3\right)-\left(4-4\sqrt{6}+6\right)$ $=8\sqrt{6}\neq 0$ $\therefore W$ is not closed under addition. W is not a subspace of $\mathbb{R}^3$.
(b)	Consider $\left(x_1,y_1,z_1\right)$ and $\left(x_2,y_2,z_2\right)\ \in V$ $\left(x_1,y_1,z_1\right)+\left(x_2,y_2,z_2\right)=\left(x_1+x_2,\ y_1+y_2,\ z_1+z_2\right)$ Since $x_1+x_2=2y_1+2y_2=2\left(y_1+y_2\right)$ and $y_1+y_2-\left(z_1+z_2\right)=y_1-z_1+\left(y_2-z_2\right)=0+0=0$ V is closed under addition. Consider $k\left(x_1,y_1,z_1\right)=\left(kx_1,ky_1,kz_1\right)$ where $k\in\mathbb{R}$ Since $kx_1=k\left(2y_1\right)=2\left(ky_1\right)$ and $ky_1-kz_1=k\left(y_1-z_1\right)=0$ V is closed under scalar multiplication Also (0, 0, 0) belongs to V $\therefore V$ is a subspace of $\mathbb{R}^3$.

2	$d=2r^2-2=2(r^2-1)$ Since S exists, $\|r\|<1$ Hence $r^2-1<0$ and $d<0$ (shown) $2r^6-2r^2=2r^2-2$ $r^6-2r^2+1=0$ Let $u=r^2$: $u^3-2u+1=0$ $(u-1)(u^2+u-1)=0$ Hence $u=1$ or $u=\dfrac{-1\pm\sqrt{5}}{2}$ Since $\|r\|<1$, $0<u<1$; hence $u=\dfrac{\sqrt{5}-1}{2}$ $d=2(r^2-1)=2(u-1)=\sqrt{5}-3$

3(a) (i)	(a) (i) Since P is a point on E, $F_1P + F_2P = 2a$ for all points P. F_1 is $(c, 0)$ and F_2 is $(-c, 0)$, where $c = \sqrt{a^2 - b^2}$ Hence the perimeter is $2a + 2\sqrt{a^2 - b^2}$
(a)(ii)	If F_1PF_2 forms an equilateral triangle, $F_1F_2 = F_1P = F_2P = a$. Hence $2ae = a \Rightarrow e = \frac{1}{2}$. Therefore equations of directrices is $x = \pm\frac{a}{\left(\frac{1}{2}\right)} = \pm 2a$
(b) (i)	$r = \frac{3}{2 + 2\sin\theta} = \frac{3/2}{1 + \sin\theta}$. Hence $e = 1$ Therefore the conic is a parabola.
(ii)	The equation of the directrix is $y = \frac{3}{2}$.

4(a)	k is the net growth rate and N is the carrying capacity of the environment.
(b)	$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{4}\right) \Rightarrow \int \frac{1}{P(4-P)}\,\mathrm{d}P = \int \frac{k}{4}\,\mathrm{d}t$ $\Rightarrow \frac{1}{4}\int\left(\frac{1}{P} + \frac{1}{4-P}\right)\mathrm{d}P = \int \frac{k}{4}\,\mathrm{d}t$ $\Rightarrow \ln P - \ln\lvert 4-P\rvert = kt + C$ where C is a real number $\Rightarrow \frac{P}{4-P} = A\mathrm{e}^{kt}$ where $A = \pm\mathrm{e}^{C}$ When $t = 0$, $P = 2 \Rightarrow A = 1$ $\Rightarrow P = (4-P)e^{kt}$ $\Rightarrow P = \frac{4\mathrm{e}^{kt}}{1+\mathrm{e}^{kt}}$
(c)	$P = \frac{4\mathrm{e}^{kt}}{1+\mathrm{e}^{kt}} = \frac{4}{\mathrm{e}^{-kt}+1}$ If $k = 0.6$, as $t \to \infty$, $\mathrm{e}^{-kt} \to 0$, so $P \to 4$ In the long run, the population of birds will increase and stabilize at 4,000.

5(a)	**[Solution]** $u_n = u_{n-1} + \frac{3}{4}u_{n-2},\ n \geq 3$ $u_1 = 40,\ \ u_2 = 50$
(b)	$u_n = u_{n-1} + \frac{3}{4}u_{n-2}$ Characteristic equation is $4m^2 - 4m - 3 = 0$ $m = -\frac{1}{2}$ or $\frac{3}{2}$ Thus, the general solution is $u_n = A\left(-\frac{1}{2}\right)^n + B\left(\frac{3}{2}\right)^n$. Since $u_1 = 40$, $\left(-\frac{1}{2}\right)A + \left(\frac{3}{2}\right)B = 40 ------(1)$ Since $u_2 = 50$, $\left(-\frac{1}{2}\right)^2 A + \left(\frac{3}{2}\right)^2 B = 50 ------(2)$ Solving by GC, $A = -10, B = \frac{70}{3}$ $u_n = -10\left(-\frac{1}{2}\right)^n + \frac{70}{3}\left(\frac{3}{2}\right)^n,\quad n \geq 1$

(c) **Total amount** save in n months

$$= u_1 + u_2 + u_3 + \dots\dots + u_n$$

$$= -10\left[\left(-\frac{1}{2}\right)^1 + \left(-\frac{1}{2}\right)^2 + \dots\dots \left(-\frac{1}{2}\right)^n\right] + \frac{70}{3}\left[\left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \dots\dots \left(\frac{3}{2}\right)^n\right]$$

$$= \frac{-10\left(-\frac{1}{2}\right)\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)} + \frac{\frac{70}{3}\left(\frac{3}{2}\right)\left(\left(\frac{3}{2}\right)^n - 1\right)}{\frac{3}{2}-1}$$

$$= \frac{10}{3}\left(1-\left(-\frac{1}{2}\right)^n\right) + 70\left(\left(\frac{3}{2}\right)^n - 1\right)$$

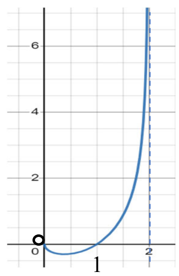
Using GC,

n	Total amount in n months
21	\$349085
22	\$523661

Hence she needs 22 months to save at least \$520 000.

6(a)	Consider $\mathbf{Av} = \lambda\mathbf{v}$ where $\mathbf{v}$ is eigenvector of $\mathbf{A}$ and λ is corresponding eigenvalue. $\begin{pmatrix} 0 & -2 & 2 \\ -k & 1 & k \\ -1 & 1 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 \\ -k+1 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\therefore -k+1 = -2 \Rightarrow k = 3$ (Shown)
(b)	$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ $\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$ $\det\begin{pmatrix} -\lambda & -2 & 2 \\ -3 & 1-\lambda & 3 \\ -1 & 1 & 3-\lambda \end{pmatrix} = 0$ $\det\begin{pmatrix} 0 & -2-\lambda & 2-3\lambda+\lambda^2 \\ -3 & 1-\lambda & 3 \\ -1 & 1 & 3-\lambda \end{pmatrix} = 0$ $-(-2-\lambda)(-9+3\lambda+3)+(2-3\lambda+\lambda^2)(-3+1-\lambda) = 0$ $(2+\lambda)(3\lambda-6)-(2-3\lambda+\lambda^2)(2+\lambda) = 0$ $(2+\lambda)(3\lambda-6-2+3\lambda-\lambda^2) = 0$ $(\lambda+2)(\lambda^2-6\lambda+8) = 0$ $(\lambda+2)(\lambda-2)(\lambda-4) = 0$ $\lambda = 2, 4 \text{ and } -2$ When $\lambda = 2$, $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$ $\begin{pmatrix} -2 & -2 & 2 \\ -3 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0}$ $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\therefore$ eigenvector $= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ When $\lambda = 4$, $(\mathbf{A} - 4\mathbf{I})\mathbf{v} = \mathbf{0}$ $\begin{pmatrix} -4 & -2 & 2 \\ -3 & -3 & 3 \\ -1 & 1 & -1 \end{pmatrix}\mathbf{v} = \mathbf{0}$

	$\begin{pmatrix}-1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-1\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\-2\\-2\end{pmatrix}$ $\therefore$ eigenvector $=\begin{pmatrix}0\\1\\1\end{pmatrix}$
	$\mathbf{D}=\begin{pmatrix}-2&0&0\\0&2&0\\0&0&4\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$
(c)	$\mathbf{B}=(\mathbf{A}-3\mathbf{I})^4$ $=(\mathbf{QDQ}^{-1}-\mathbf{Q}3\mathbf{IQ}^{-1})^4$ $=(\mathbf{Q}(\mathbf{D}-3\mathbf{I})\mathbf{Q}^{-1})^4$ $=(\mathbf{Q}(\mathbf{D}-3\mathbf{I})\mathbf{Q}^{-1})(\mathbf{Q}(\mathbf{D}-3\mathbf{I})\mathbf{Q}^{-1})(\mathbf{Q}(\mathbf{D}-3\mathbf{I})\mathbf{Q}^{-1})(\mathbf{Q}(\mathbf{D}-3\mathbf{I})\mathbf{Q}^{-1})$ $=\mathbf{Q}(\mathbf{D}-3\mathbf{I})^4\mathbf{Q}^{-1}$ Hence $\mathbf{P}=\mathbf{Q}=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$ $\mathbf{C}=(\mathbf{D}-3\mathbf{I})^4=\begin{pmatrix}-5&0&0\\0&-1&0\\0&0&1\end{pmatrix}^4=\begin{pmatrix}625&0&0\\0&1&0\\0&0&1\end{pmatrix}$

7(a)	As $\theta \to \frac{\pi}{2}, r\cos\theta \to 2$ and $r\sin\theta \to \infty$. Hence there is a vertical asymptote $x = 2$.
(b)	
(c)	When $y = 0$, $r\sin\theta = 0 \Rightarrow \theta = 0$. $\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{0} (\tan\theta+\sec\theta)^2 \; \mathrm{d}\theta$ $= \frac{1}{2}\int_{-\frac{\pi}{2}}^{0} \left(\tan^2\theta + 2\tan\theta\sec\theta + \sec^2\theta\right) \mathrm{d}\theta$ $= \frac{1}{2}\int_{-\frac{\pi}{2}}^{0} \left((\sec^2\theta - 1) + 2\tan\theta\sec\theta + \sec^2\theta\right) \mathrm{d}\theta$ $= \frac{1}{2}\left[2(\tan\theta + \sec\theta) - \theta\right]_{-\frac{\pi}{2}}^{0}$ $= \frac{1}{2}\left(2 - \frac{\pi}{2}\right) = 1 - \frac{\pi}{4}$ units^2
(d)	$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \sec^2\theta + \sec\theta\tan\theta$ $\text{Arc length} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$ $= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{(\tan\theta + \sec\theta)^2 + (\sec^2\theta + \sec\theta\tan\theta)^2}\,\mathrm{d}\theta$ $= 2.67$ (3sf)

8(a)	$\mathbf{A}=\begin{pmatrix}1 & -2 & 1 & -1\\ 2 & -3 & 2 & \alpha-6\\ 1 & 5\alpha-2 & 1 & -1\\ 0 & 1 & 0 & \alpha-4\end{pmatrix}$ $\xrightarrow{R_2:-2R_1+R_2}\begin{pmatrix}1 & -2 & 1 & -1\\ 0 & 1 & 0 & \alpha-4\\ 1 & 5\alpha-2 & 1 & -1\\ 0 & 1 & 0 & \alpha-4\end{pmatrix}$ $\xrightarrow[R_4:-R_2+R_4]{R_3:-R_1+R_3}\begin{pmatrix}1 & -2 & 1 & -1\\ 0 & 1 & 0 & \alpha-4\\ 0 & 5\alpha & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ $\xrightarrow{R_3:-5\alpha R_2+R_3}\begin{pmatrix}1 & -2 & 1 & -1\\ 0 & 1 & 0 & \alpha-4\\ 0 & 0 & 0 & -5\alpha(\alpha-4)\\ 0 & 0 & 0 & 0\end{pmatrix}$ Since dim of null space of T = 1, Rank($\mathbf{A}$) = 4 – 1 = 3 $-5\alpha(\alpha-4)\neq 0$ $\therefore \alpha \neq 0, 4, \quad \alpha\in\mathbb{R}$
(b)	When $\alpha=0$, we have $\begin{pmatrix}1 & -2 & 1 & -1\\ 0 & 1 & 0 & -4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ A basis for range space of T is $\left\{\begin{pmatrix}1\\2\\1\\0\end{pmatrix},\begin{pmatrix}-2\\-3\\-2\\1\end{pmatrix}\right\}$.

(c)	Since $\begin{pmatrix} p \\ -7 \\ q \\ 3 \end{pmatrix}$ lies in the range space of T, $a\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}+b\begin{pmatrix} -2 \\ -3 \\ -2 \\ 1 \end{pmatrix}=\begin{pmatrix} p \\ -7 \\ q \\ 3 \end{pmatrix}$ has unique solution for a and b. Since $b = 3$ and $a = 1$, hence solving the equations we get $p=q=-5$
(d)	$\begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $x-2y+z-w=0$ $y-4w=0$ $\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}=z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}+w\begin{pmatrix} 9 \\ 4 \\ 0 \\ 1 \end{pmatrix}$ Since $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\neq k\begin{pmatrix} 9 \\ 4 \\ 0 \\ 1 \end{pmatrix}$ for any $k\in\mathbb{R}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 4 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent. Hence a basis for the null space of T is $\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\begin{pmatrix} 9 \\ 4 \\ 0 \\ 1 \end{pmatrix}\right\}$
(e)	$\mathbf{Ax}=\begin{pmatrix} -2 \\ -3 \\ -2 \\ 1 \end{pmatrix}$ has a particular solution $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Thus the general solution is $\mathbf{x}=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}+\mu\begin{pmatrix} 9 \\ 4 \\ 0 \\ 1 \end{pmatrix}$ where $\lambda,\mu\in\mathbb{R}$

9(a)	$I_n=\int_0^{\frac{\pi}{4}}\tan^n x\ \mathrm{d}x=\int_0^{\frac{\pi}{4}}\tan^{n-2}x.\tan^2 x\ \mathrm{d}x$ $=\int_0^{\frac{\pi}{4}}\tan^{n-2}x.(\sec^2 x-1)\ \mathrm{d}x$ $=\left[\frac{1}{n-1}\tan^{n-1}x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan^{n-2}x\ \mathrm{d}x$ $=\frac{1}{n-1}-I_{n-2}$ (shown) $I_1=\int_0^{\frac{\pi}{4}}\tan x\ \mathrm{d}x=\left[\ln(\sec x)\right]_0^{\frac{\pi}{4}}=\ln\sqrt{2}=\frac{1}{2}\ln 2$ $I_3=\frac{1}{3-1}-I_1$ $=\frac{1}{2}-\frac{1}{2}\ln 2$
(b)	Using shell method, Volume = $\pi\left(\frac{5\pi}{2}\right)^2(5\pi)-2\pi\int_0^{\frac{5\pi}{2}}x^2(\sin x+1)\ \mathrm{d}x$ $=\frac{125\pi^4}{4}-2\pi\int_0^{\frac{5\pi}{2}}x^2\sin x\ \mathrm{d}x-2\pi\int_0^{\frac{5\pi}{2}}x^2\ \mathrm{d}x$ $2\pi\int_0^{\frac{5\pi}{2}}x^2\ \mathrm{d}x=2\pi\left[\frac{x^3}{3}\right]_0^{\frac{5\pi}{2}}=\frac{125\pi^4}{12}$ $2\pi\int_0^{\frac{5\pi}{2}}x^2\sin x\ \mathrm{d}x=2\pi\left(\left[-x^2\cos x\right]_0^{\frac{5\pi}{2}}+\int_0^{\frac{5\pi}{2}}2x\cos x\ \mathrm{d}x\right)$ $=2\pi\left(0+\left[2x\sin x\right]_0^{\frac{5\pi}{2}}-\int_0^{\frac{5\pi}{2}}2\sin x\ \mathrm{d}x\right)$ $=2\pi\left(5\pi+\left[2\cos x\right]_0^{\frac{5\pi}{2}}\right)$ $=2\pi(5\pi-2)$ Hence required volume = $\frac{125\pi^4}{4}-2\pi(5\pi-2)-\frac{125\pi^4}{12}=\frac{125\pi^4}{6}-10\pi^2+4\pi$ units^3

10(a)	Number of ways $=\binom{6}{4}4!=360$
(b)	**Case 1 : All 4 are of different flavours** No. of ways = 360 **Case 2 : 2 are of the same flavour, the remaining 2 are different flavours** No. of ways $=\binom{2}{1}\binom{5}{2}\frac{4!}{2!}=240$ **Case 3 : 3 of them have the same flavour (durian) and the last one has a different flavour** No. of ways $=\binom{1}{1}\binom{5}{1}\frac{4!}{3!}=20$ **Case 4 : 2 are of the same flavour, the other 2 have the same flavour (i.e. two durian and 2 chocolate)** No. of ways $=\binom{2}{2}\frac{4!}{2!2!}=6$ **Case 5 : All 4 are of the same durian flavour.** No. of ways $=1$ Total number of ways = 627
(c)	Number of ways $=\frac{5!}{5}\times 2!\times 4!=$ 1152 where: Circular arrangement (PCCCCP is one unit and the other 4 people form the 5 units around a circle) – $\frac{5!}{5}$ Arrange the 4 children – 4! Arrange the parents – 2!