# Differential Equations I [part FM]

**1** **(i)** By means of the substitution $y = vx$, show that the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2, \ \ x \neq 0$$

can be reduced to $\dfrac{\mathrm{d}v}{\mathrm{d}x} = -\dfrac{v^2+1}{2vx}$. [3]

**(ii)** Hence find the general solution. [4]

**(iii)** Sketch the solution curve passing through $(2, 0)$. [2]

[SRJC/Promo/2016/9]

**2** Find the general solution of the differential equation $\cos t\dfrac{\mathrm{d}y}{\mathrm{d}t} - y\sin t = \cos t$, where $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$, giving your answer in the form $y = \mathrm{f}(t)$. [3]

On a single diagram, sketch

**(i)** the solution curve that passes through the origin, [1]

**(ii)** the solution curve which has a minimum point when $t = -\frac{1}{6}\pi$. [4]

[AJC/Prelim/2005/I/15(b)OR]

**3** **(a)** Find the general solution of the differential equation $\left(1+\cos\theta\right)\dfrac{\mathrm{d}x}{\mathrm{d}\theta} + x = 1$. [5]

**(b)** Show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 2y^2\ln x$$

may be reduced by the substitution $y = \dfrac{1}{u}$ to

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{x} = -2\ln x\,.$$

Hence find the general solution for $y$ in terms of $x$. [5]

[MJC/Prelim/2005/I/11]

**4** **(a)** **(i)** Given that $x > 0$, find the general solution of the differential equation

$$x^2 \frac{dy}{dx} = y,$$

expressing $y$ as a function of $x$. [3]

**(ii)** Sketch the graph of the particular solution for which $y \to 2$ as $x \to \infty$. [3]

**(b)** Water starts pouring into an empty open tank, and $t$ seconds later the volume, $V$ litres, of water in the tank is given by $\frac{dV}{dt} + \frac{1}{20}V = \frac{1}{4}t + 1$.

**(i)** Find $V$ in terms of $t$, and hence show that, for large values of $t$, $V \approx 5t - 80$. [5]

**(ii)** What can we deduce from **(b)(i)**, in the context of this question? [1]

**5** **(a)** Find the general solution of the differential equation

$$(1+x)y - x\frac{dy}{dx} = x^3 - x^2,$$

leaving your answer in the form $y = f(x)$. [5]

Sketch and label clearly the equations of 2 **distinct** members of the family of solution curves where their nature of stationary points differ from each other. [2]

**(b)** It is given that $\frac{dN}{dt} = 3t - N^2$ and $N(0) = 1$. Use two iterations of the improved Euler's method to find an approximate value of $N(0.4)$, correct to 4 decimal places. [4]

[NJC/H2 FM/2017/Prelim/P2/Q4]

**6** It is given that $y = f(x)$ is a particular solution of the differential equation

$$\frac{dy}{dx} = 2x - y^2$$

with the initial condition $f(0) = 1$.

**(i)** Use the Euler method with two steps of equal size, starting at $x = 0$, to approximate $f(-0.2)$. [2]

**(ii)** Find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$. [1]

Determine whether the approximation found in **(i)** is an under-estimate or an over-estimate. [2]

[RI/H2 FM/2017/Prelim/P1/Q5b]

**7** An industrial cooler initially contains 50 litres of pure water. When the temperature of the water rises to 60 ºC, a coolant containing 30 grams of chemical $X$ in every litre of water is added into the cooler at a constant rate of 4 litres per minute. The mixture is instantaneously mixed thoroughly and flows out of the cooler at a constant rate of 3 litres per minute through another outlet.

**(i)** If $x$ grams is the amount of chemical $X$ in the cooler $t$ minutes after the coolant is added, show that $x$ satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 120 - \frac{3x}{50+t}.$$ [2]

**(ii)** Find the amount of chemical $X$ in the cooler when there are 51 litres of mixture in it. [6]

The temperature of the mixture, $\theta$ ºC follows the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 60\mathrm{e}^{0.04t}\left[\cos(0.01\theta) - 1\right].$$

Use the improved Euler method with step size 0.5 to estimate to 1 decimal place, the temperature of the mixture 1 minute after the start of the addition of coolant. [4]

[VJC/H2 FM/2017/Prelim/P1/Q11]

| **S/N** | **Answers** |
|---|---|
| **1** | **(ii)** $y^2 + x^2 = Ax$ **(iii)** $y^2 + (x-1)^2 = 1$ |
| **2** | $y = \tan t + c\sec t$ **(i)** $y = \tan t$ **(ii)** $y = \tan t + 2\sec t$ |
| **3** | **(a)** $x = 1 + c\mathrm{e}^{-\tan\frac{\theta}{2}}$ **(b)** $y = \dfrac{1}{Cx - x(\ln x)^2}$ |
| **4** | **(a)(i)** $y = B\mathrm{e}^{-\frac{1}{x}}$ **(ii)** $y = 2\mathrm{e}^{-\frac{1}{x}}$ <br> **(b)(i)** $V = 5t - 80 + C\mathrm{e}^{-\frac{1}{20}t}$ |
| **5** | **(a)** $y = x^2 + Cx\mathrm{e}^x$ <br> **(b)** 0.9225 (4 d.p.) |
| **6** | **(i)** $\mathrm{f}(-0.2) = 1.241$ **(ii)** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 - 4xy + 2y^3$ |
| **7** | **(ii)** 117 grams (3 s.f.) ; 50.9 ºC (1 d.p.). |