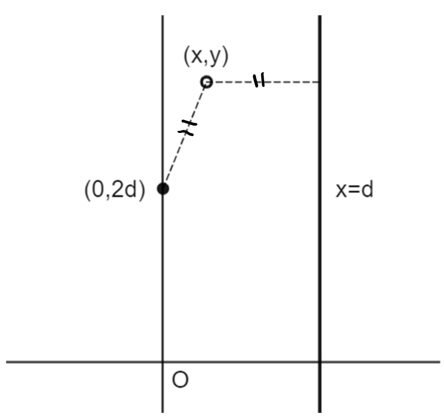
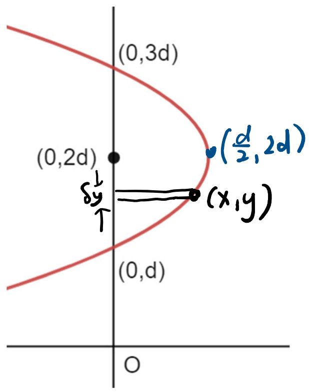
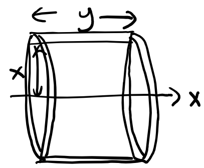
1(a)	$w^3 = 1$ $w^3 - 1 = 0$ $(w-1)(w^2+w+1) = 0$ Since w is non-real, it satisfies $w^2+w+1=0$. $\therefore w^2 + w = -1$.	- GC is not allowed and given 2 marks, the intent is not to solve for *w*.
(b)	$(1+2w+3w^2)(1+2w^2+3w) = 3$ LHS: $\left[1+2(w+w^2)+w^2\right]\left[1+2(w^2+w)+w\right]$ $=\left[1+2(-1)+w^2\right]\left[1+2(-1)+w\right]$ $=\left(w^2-1\right)\left(w-1\right)$ $= w^3-(w^2+w)+1$ $=1-(-1)+1$ $=3$ **Alternative** LHS $= (1+2w+3w^2)+(2w^2+4w^3+6w^4)+(3w+6w^2+9w^3)$ $=1+5w+11w^2+13w^3+6w^4$ $=1+5w+11w^2+13+6w \quad \left(\because w^3=1 \Rightarrow w^4=w\right)$ $=14+11w+11w^2$ $=14+11(-1)$ $=3$ $=$ RHS	- Make good use of result in (a) to simplify. - Note the efficient way to evaluate w^4.
2(a)	 distance from (x,y) to $(0,2d) = \sqrt{x^2+(y-2d)^2}$ distance from (x,y) to line $= \lvert x-d \rvert$ $\sqrt{x^2+(y-2d)^2} = \lvert x-d \rvert$ $x^2+(y-2d)^2 = (x-d)^2$ $(y-2d)^2 = (x-d)^2 - x^2$ $= (x-d-x)(x-d+x)$ $= -d(2x-d)$ $(y-2d)^2 = d(d-2x)$ (shown)	From definition, $d > x$. $\lvert x-d \rvert = d-x$

(b)





By shell method,

$\delta A \approx 2\pi xy\delta y$

$(y-2d)^2 = d(d-2x)$

$\Rightarrow d - 2x = \dfrac{(y-2d)^2}{d}$

$\Rightarrow x = \dfrac{d}{2} - \dfrac{(y-2d)^2}{2d}$

$$\text{Volume} = 2\pi\int_d^{3d} xy \, dy$$

$$= 2\pi\int_d^{3d} xy \, dy$$

$$= 2\pi\int_d^{3d}\left(\frac{d}{2} - \frac{(y-2d)^2}{2d}\right)y \, dy$$

$$= \pi d\int_d^{3d} y \, dy - \frac{\pi}{d}\int_d^{3d} y(y-2d)^2 \, dy$$

$$= \pi d\left[\frac{y^2}{2}\right]_d^{3d} - \frac{\pi}{d}\left[\int_d^{3d}(y-2d)(y-2d)^2 + 2d(y-2d)^2 \, dy\right]$$

$$= \frac{\pi d}{2}\left(9d^2 - d^2\right) - \frac{\pi}{d}\left[\frac{(y-2d)^4}{4}\right]_d^{3d} - 2\pi\left[\frac{(y-2d)^3}{3}\right]_d^{3d}$$

$$= 4\pi d^3 - \frac{\pi}{4d}\left(d^4 - d^4\right) - \frac{2\pi}{3}\left(d^3 + d^3\right)$$

$$= \frac{8}{3}\pi d^3 \text{ cu. units}$$

Alternative: Expand $y(y-2d)^2$; the method shown is simpler to evaluate limits.

Alternative (Disc Method)

$(y-2d)^2 = d(d-2x) \Rightarrow y = 2d \pm \sqrt{d(d-2x)}$

Required volume

= volume generated by upper curve y_U – volume generated by lower curve y_L

$$= \int_0^{\frac{d}{2}} \pi y_u^2 dx - \int_0^{\frac{d}{2}} \pi y_L^2 dx$$

$$= \pi\int_0^{\frac{d}{2}}\left[2d + \sqrt{d(d-2x)}\right]^2 - \left[2d - \sqrt{d(d-2x)}\right]^2 dx$$

$$= \pi\int_0^{\frac{d}{2}} 8d\sqrt{d(d-2x)} \, dx$$

$$= 8\pi d \times \frac{1}{-2d}\int_0^{\frac{d}{2}} -2d\sqrt{d(d-2x)} \, dx$$

$$= -4\pi \left.\frac{\left[d(d-2x)\right]^{3/2}}{3/2}\right|_0^{\frac{d}{2}}$$

$$= -\frac{8\pi}{3}\left[0 - d^3\right]$$

$$= \frac{8\pi}{3}d^3 \text{ cu. units}$$

3(a)

$$\frac{2k}{2k+1}-\frac{2k+2}{2k+3}=\frac{2k(2k+3)-(2k+2)(2k+1)}{(2k+1)(2k+3)}$$

$$=\frac{4k^2+6k-(4k^2+6k+2)}{(2k+1)(2k+3)}$$

$$=\frac{-2}{(2k+1)(2k+3)}<0 \quad (\because (2k+1)\ \&\ (2k+3)>0 \text{ for } k>0)$$

$$\therefore \frac{2k}{2k+1}<\frac{2k+2}{2k+3}$$

(b)

Let P_n be the proposition that $u_n \le \left(\frac{2n}{2n+1}\right)^n$, $n \in \mathbb{Z}^+$..

Consider P_1:

LHS $= u_1 = \frac{2}{3}$ (given)

RHS $= \frac{2(1)}{2(1)+1} = \frac{2}{3}$

Hence $u_1 \le \frac{2}{3}$

$\therefore P_1$ is true

Assume that P_k is true for **some** positive integer k, $u_k \le \left(\frac{2k}{2k+1}\right)^k$.

To show P_{k+1} is true, $u_{k+1} \le \left(\frac{2k+2}{2k+3}\right)^k$

LHS of P_{k+1}

$$= u_{k+1}$$

$$= \frac{2k+2}{2k+3}u_k$$

$$\le \left(\frac{2k+2}{2k+3}\right)\left(\frac{2k}{2k+1}\right)^k$$

$$\le \left(\frac{2k+2}{2k+3}\right)\left(\frac{2k+2}{2k+3}\right)^k \quad \text{since } \frac{2k}{2k+1}<\left(\frac{2k+2}{2k+3}\right)$$

$$= \left(\frac{2k+2}{2k+3}\right)^{k+1}$$

$$\therefore u_{k+1} \le \left(\frac{2k+2}{2k+3}\right)^{k+1}$$

$\therefore$ P_k is true $\Rightarrow P_{k+1}$ is true

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true,
by mathematical induction, P_n is true for all positive integers n.

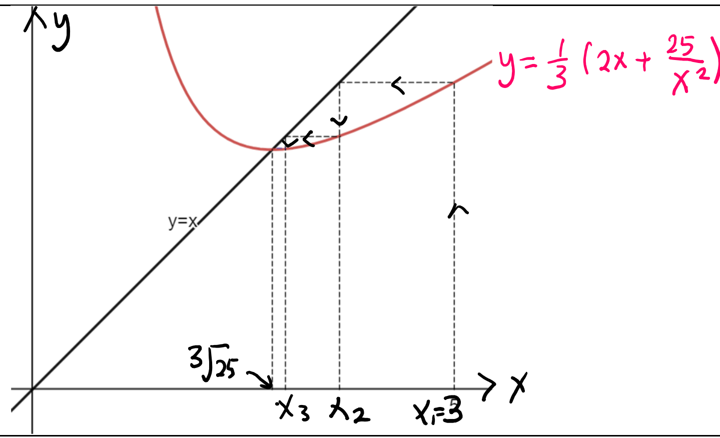
$\Rightarrow$ implies
$\rightarrow$ tends to
$\mapsto$ maps to

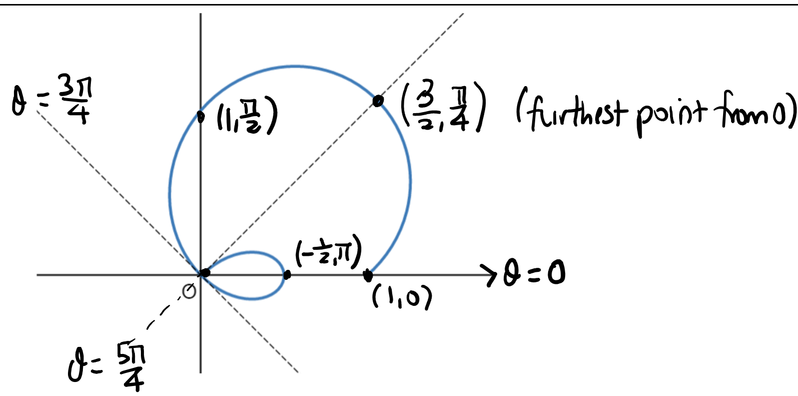
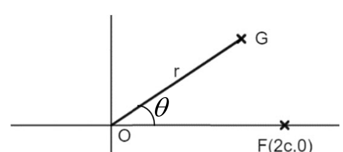
4(a)

$f(x) = x^k - a \quad \Rightarrow f'(x) = kx^{k-1}$

By Newton-Raphson method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\therefore x_{n+1} = x_n - \frac{{x_n}^k - a}{k{x_n}^{k-1}}$$

	$x_{n+1}=x_n-\dfrac{x_n^{\,k}}{kx_n^{\,k-1}}+\dfrac{a}{kx_n^{\,k-1}}$ $=x_n-\dfrac{x_n}{k}+\dfrac{a}{kx_n^{\,k-1}}$ $=\dfrac{1}{k}\left[(k-1)x_n+\dfrac{a}{x_n^{\,k-1}}\right]$ (shown)	A1
(b)	Since $\sqrt[3]{25}\approx\sqrt[3]{27}=3,\ 3$ is a reasonable initial estimate.	B1
(c)	Using $x_1=3,$ from GC, $x_2=2.92592...$ $x_3=2.924018...$ $x_4=2.924017...$ $x_5=2.924017...$ $\therefore \sqrt[3]{23}=2.924$ (3 dp)	M1 A1
(d)		Question stated to sketch suitable graphs (hence more than 1), so the graphs of $y=x$ and $y=\dfrac{1}{3}\left(2x+\dfrac{25}{x^2}\right)$ are expected
5	$v_{n+2}-2v_{n+1}+2v_n=4,\quad v_1=2,\ v_2=0$ $v_n+w_n=k\Rightarrow v_n=k-w_n$ $v_{n+2}-2v_{n+1}+2v_n=4$ becomes $(k-w_{n+2})-2(k-w_{n+1})+2(k-w_n)=4$ $\therefore w_{n+2}-2w_{n+1}+2w_n=k-4$ For this equation to be homogeneous, $k=4$. $w_1=4-v_1=2, w_2=4-v_2=4.$ $w_{n+2}-2w_{n+1}+2w_n=0$ Auxiliary equation is $\lambda^2-2\lambda+2=0$ $\lambda=\dfrac{2\pm\sqrt{4-8}}{2}=1\pm \mathrm{i}$ $\therefore w_n=\left(\sqrt{2}\right)^n\left[A\cos\dfrac{n\pi}{4}+B\sin\dfrac{n\pi}{4}\right]$ $w_1=2\Rightarrow 2=A+B$ $v_2=4\Rightarrow 4=2(0+B)$ $\Rightarrow B=2$ $\Rightarrow A=0$ $\therefore w_n=\left(\sqrt{2}\right)^n\left[2\sin\dfrac{n\pi}{4}\right]$ $\Rightarrow v_n=4-2^{\frac{n}{2}+1}\sin\dfrac{n\pi}{4}$	It is totally unthinking to write auxiliary equation as: $\lambda^2-2\lambda+2=4.$

(b)	$v_{4n+2} = 4 - 2^{\frac{4n+2}{2}+1}\sin\frac{(4n+2)\pi}{4}$ $= 4 - 2^{2n+2}\sin\left(2n+1\right)\frac{\pi}{2}$ $= \begin{cases} 4 - 2^{2n+2}(-1), & n \text{ is odd} \\ 4 - 2^{2n+2}(1), & n \text{ is even} \end{cases}$ $= \begin{cases} 4(1+4^n), & n \text{ is odd} \\ 4(1-4^n), & n \text{ is even} \end{cases}$	
6(a)	<table><tr><td>x</td><td>$\frac{3\pi}{4}$</td><td>π</td><td>$\frac{5\pi}{4}$</td></tr><tr><td>y^2</td><td>0</td><td>$\frac{1}{4}$</td><td>0</td></tr></table> $\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} y^2 \, dx \approx \frac{\frac{5\pi}{4} - \frac{3\pi}{4}}{6}\left[0+0+4\left(\frac{1}{4}\right)\right] = \frac{1}{12}\pi \approx 0.262$ (3sf)	
(b)	 Area of the inner loop $= \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} r^2 \, d\theta$ $\approx \frac{1}{2}\left(\frac{\pi}{12}\right) = \frac{\pi}{24} = 0.131$ (3sf)	
7(a)	Locus is an ellipse.	
(b)	 $OG + GF = 2a$ $r + \sqrt{r^2 + 4c^2 - 4rc\cos\theta} = 2a$ (use cosine rule to find GF) $\sqrt{r^2 + 4c^2 - 4rc\cos\theta} = 2a - r$ $r^2 + 4c^2 - 4rc\cos\theta = 4a^2 - 4ar + r^2$ $4ar\left(1 - \frac{c}{a}\cos\theta\right) = 4(a^2 - c^2)$ $r = \frac{a - \frac{c^2}{a}}{\left(1 - \frac{c}{a}\cos\theta\right)}$ (shown) $\therefore p = a - \frac{c^2}{a}$ and $q = \frac{c}{a}$.	A1 p A1 q

(c)	q represents the eccentricity of the conic section.	
(d)	Arc length = $\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ $= \int_0^{\pi} \sqrt{\left(\frac{a - \frac{c^2}{a}}{1 - \frac{c}{a}\cos\theta}\right)^2 + \frac{\left(\frac{c^2}{a} - a\right)^2 \left(\frac{c}{a}\sin\theta\right)^2}{\left(1 - \frac{c}{a}\cos\theta\right)^4}}\, d\theta$ For small values of c, the ellipse approaches the circle with centre at O and radius a. $\therefore$ arc length $\approx \pi a$. **Alternative:** $\lim_{c \to 0} \sqrt{\left(\frac{a - \frac{c^2}{a}}{1 - \frac{c}{a}\cos\theta}\right)^2 + \frac{\left(\frac{c^2}{a} - a\right)^2 \left(\frac{c}{a}\sin\theta\right)^2}{\left(1 - \frac{c}{a}\cos\theta\right)^4}} = \sqrt{\frac{a^2}{1} + \frac{a^2 \times 0}{1}} = a$ $\therefore$ arc length $\to \int_0^{\pi} a\, d\theta = a\pi$	
8(a)	Number of ways $= 12! = 479001600$	
(b)	WWWWW MMMMM MMMMMMM WWWWWMM Number of ways = $5! \times 7! + {}^{7}P_5 \times 5! \times 2! \times 3 = 2419200$	
(c)	Consider complementary cases involving two particular girls are adjacent to each other. Case 1: Both seated in front row $2! \times {}^{10}C_3 \times 4! \times 7! = 29030400$ Case 2: Both seated in back row $2! \times {}^{10}C_5 \times 6! \times 5! = 43545600$ Number of ways $= 479001600 - (29030400 + 43545600)$ $= 406425600$	
(d)	Number of ways = $\frac{{}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4}{3!} = 5775$	
9 (a)	$\overrightarrow{AB} = \begin{pmatrix} -6 \\ -4 \\ 0 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix}$ $\underset{\sim}{n} = \begin{pmatrix} -6 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -24 \end{pmatrix} // \begin{pmatrix} 2 \\ -3 \\ 12 \end{pmatrix}$ $\therefore$ equation of plane ABC is $\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 12 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 12 \end{pmatrix} = 12$ Cartesian equation is $2x - 3y + 12z = 12$.	

(b)	Consider line OD: $\underset{\sim}{r} = \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$ When OD intersects plane ABC, $2(2\lambda) - 3(0) + 12(3\lambda) = 12.$ $\lambda = 0.3$ $\therefore$ point of intersection is (0.6, 0, 0.9).	Answer in coordinates form, not position vector.
(c)	Shortest distance = $\left\| \overrightarrow{AD} \cdot \hat{\underset{\sim}{n}} \right\| = \frac{1}{\sqrt{4+9+144}} \left\| \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 12 \end{pmatrix} \right\| = \frac{108}{\sqrt{157}}$ units.	
(d)	Area of plane $ABC = \frac{1}{2}\left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \frac{1}{2}\sqrt{628} = \sqrt{157}$ $\therefore$ Volume of tetrahedron = $\frac{1}{3} \times \sqrt{157} \times \frac{108}{\sqrt{157}} = 36$ cu. units.	
10(a)	$x = at^2, \quad y = 2at, \quad -\sqrt{a} \le t \le \sqrt{a}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 2at, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2a$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{t}$ Gradient of $TS = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_P = \frac{1}{t}$ $\therefore \tan\theta =$ gradient of $TS = \frac{1}{t}$ (shown) Gradient of $QP = \frac{2at-0}{at^2-a} = \frac{2t}{t^2-1}$ $\therefore \tan\phi = \frac{2t}{t^2-1}$ $\tan\phi = \frac{\frac{2}{\tan\theta}}{\frac{1}{\tan^2\theta} - 1}$ $= \frac{2\tan^2\theta}{\tan\theta(1-\tan^2\theta)}$ $= \frac{2\tan\theta}{1-\tan^2\theta}$ $= \tan 2\theta$ Since $0 < \phi, \theta < \pi, \quad \phi = 2\theta$ (shown)	When tan A = tan B, there are many possible relationships between A and B. A good answer should include justifying why $A = B$.
(b)	$\angle QPR = \pi - 2\theta$ (corresponding angles) $\angle TPQ = \pi - (\pi - 2\theta) - \theta$ (sum of angles on a straight line) $\therefore \angle TPQ = \theta$ (shown)	
(c)	The inner reflective surface of the car headlight is to have a <u>parabolic</u> shape with the <u>light source located at point Q</u> (focus) so that light from Q is reflected parallel to the road.	2 marks – one each to the underlined factors.

(d)	Surface area $= 2\pi\int_0^{\sqrt{a}} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$ $= 2\pi\int_0^{\sqrt{a}} 2at\sqrt{4a^2t^2+4a^2}\ \mathrm{d}t$ $= 8\pi a^2\int_0^{\sqrt{a}} t\sqrt{t^2+1}\ \mathrm{d}t$ $= 4\pi a^2\left[\frac{2(t^2+1)^{3/2}}{3}\right]_0^{\sqrt{a}}$ $= \frac{8\pi a^2}{3}\left[(a+1)^{3/2}-1\right]$	Only need to consider the upper half (or lower half) of the curve.
11(a)	D_1 is always 30 m above ground.	
(b)	When $\mathbf{r}_1=\mathbf{r}_2$, $\begin{pmatrix}0\\0\\30\end{pmatrix}+t\begin{pmatrix}3\\-2\\0\end{pmatrix}=\begin{pmatrix}10\\8\\0\end{pmatrix}+t\begin{pmatrix}-1\\2\\3\end{pmatrix}$ $3t=10-t\cdots(1)$ $-2t=8+2t\cdots(2)$ $30=3t\cdots(3)$ $(1)\Rightarrow t=2.5,\quad (2)\Rightarrow t=-2\quad (3)\Rightarrow t=10$ Since there is no consistent value of t, $\mathbf{r}_1\neq\mathbf{r}_2$ at all times. $\therefore D_1$ and D_2 will not coincide. $\overrightarrow{D_1D_2}=\begin{pmatrix}10\\8\\-30\end{pmatrix}+t\begin{pmatrix}-4\\4\\3\end{pmatrix}$ $\left\lvert\overrightarrow{D_1D_2}\right\rvert=\sqrt{(10-4t)^2+(8+4t)^2+(3t-30)^2}$ $=\sqrt{1064-196t+41t^2}$ $=\sqrt{41\left(t-\frac{196}{2(41)}\right)^2+1064-41\left(\frac{98}{41}\right)^2}$ $=\sqrt{41\left(t-\frac{98}{41}\right)^2+\frac{34020}{41}}\geq\sqrt{\frac{34020}{41}}$ Minimum distance occurs when $t=\frac{98}{41}$. <u>Alternative:</u> Minimum distance occurs when $\frac{\mathrm{d}}{\mathrm{d}t}(1064-196t+41t^2)=0$ $\Rightarrow 82t=196$ $\Rightarrow t=\frac{98}{41}$ $\therefore$ minimum $\left\lvert\overrightarrow{D_1D_2}\right\rvert=\sqrt{\frac{34020}{41}}=28.8$ m (3sf)	d

(c)	Angle of elevation = angle between path of D_2 and ground $=90^\circ-\cos^{-1}\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}\right\|}{\sqrt{14}}$ $=90-\cos^{-1}\dfrac{3}{\sqrt{14}}$ $=53.3^\circ$	
(d)	At $t=7, \mathbf{r}_1=\begin{pmatrix}21\\-14\\30\end{pmatrix}$ and $\mathbf{r}_2=\begin{pmatrix}3\\22\\21\end{pmatrix}$. $\therefore h_1=30, h_2=21$. $A_1=2(30)^2, A_2=2(21)^2$. Centre of these circular areas are $(21,-14)$ and $(3, 22)$ on the $x-y$ plane with radii $R_1=\sqrt{\dfrac{A_1}{\pi}}=30\sqrt{\dfrac{2}{\pi}}$ $R_2=21\sqrt{\dfrac{2}{\pi}}$ respectively. Distance between the 2 centres $=\sqrt{(21-3)^2+(-14-22)^2}\approx 40.249$ $R_1+R_2\approx 40.692>$ distance between the 2 centres, the areas observed by both drones overlapped. (shown)