2007 H2 Mathematics Prelim Paper 1 solutions

	Solution
1.	$1, \frac{4}{(2-x)^3}$ $= 4(2^{-3})(1-\frac{x}{2})^{-3}$ $= \frac{1}{2}\left(1-3\left(-\frac{x}{2}\right)+\frac{(-3)(-4)}{2!}\left(-\frac{x}{2}\right)^2+....\right)$ $= \frac{1}{2}+\frac{3}{4}x + \frac{3}{4}x^2+...$ for $\lvert x \rvert < 2$ Coefficient of x^n $= \left(\frac{1}{2}\right)\left[\frac{(-3)(-4)...(-[n+2])}{n!}\left(-\frac{1}{2}\right)^n\right]$ $= \left(\frac{1}{2}\right)(-1)^{2n}\frac{(n+1)(n+2)}{(1)(2)}\left(\frac{1}{2}\right)^n$ $= \left(\frac{1}{2}\right)^{n+2}(n+1)(n+2)$
2.	$y = (\cos^{-1}x)^2$ $\frac{dy}{dx} = 2\cos^{-1}x\left(-\frac{1}{\sqrt{1-x^2}}\right)$ $\left(\sqrt{1-x^2}\right)\frac{dy}{dx} = -2\cos^{-1}x$ $(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\cos^{-1}x)^2 = 4y$ (proved)
(i)	Differentiating wrt x , $(1-x^2)2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - 2x\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$ $(1-x^2)\left(\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right) = 2$ When $x = 0$, $y = (\cos^{-1}0)^2 = \frac{\pi^2}{4}$; $\frac{dy}{dx} = -\pi$; $\frac{d^2y}{dx^2} = 2$; By Maclaurin's Theorem , $y = \frac{\pi^2}{4} - \pi x + x^2 + ...$
(ii)	At x = 0, equation of tangent to the curve is $y = \frac{\pi^2}{4} - \pi x$
3a)	a) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \lvert\sin x\rvert\, dx = \int_{-\frac{\pi}{4}}^{0} \sin x(-\sin x)\, dx + \int_{0}^{\frac{\pi}{2}} \sin x(\sin x) dx$

	Solution
	$= -\int_{-\frac{\pi}{4}}^{0} \frac{1-\cos 2x}{2}\,dx + \int_{0}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx$ $= -\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_{-\frac{\pi}{4}}^{0} + \frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_{0}^{\frac{\pi}{2}} = = \frac{\pi}{8}+\frac{1}{4}$
b)	$x=\frac{1}{u} \quad \Rightarrow \quad \frac{dx}{du}=-\frac{1}{u^2}$ $\int \frac{1}{\mathrm{x}\sqrt{\mathrm{x}^2-2}}\,\mathrm{d}x = \int \frac{u}{\sqrt{\frac{1}{u^2}-2}}\,(-\frac{1}{u^2})\,du$ $= \int \frac{-1}{\sqrt{1-2u^2}}\,du$ $= -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}u + c \quad = \quad c-\frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{2}}{x})$ *Alternately*: $\frac{1}{\sqrt{2}}\cos^{-1}(\frac{\sqrt{2}}{x})+c$
4. (i)	Direction vector of ℓ_1 is $\begin{pmatrix}0\\-2\\5\end{pmatrix}$ Equation of ℓ_1 is $\underset{\sim}{r} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\-2\\5\end{pmatrix}$, $\lambda \in \mathbb{R}$
(ii)	$\overrightarrow{OB} = \begin{pmatrix}4\\13\\-3\end{pmatrix}$ & $\overrightarrow{ON} = \begin{pmatrix}2\\-1-2\lambda\\3+5\lambda\end{pmatrix}$ Then $\overrightarrow{BN} = \overrightarrow{ON} - \overrightarrow{OB} = \begin{pmatrix}-2\\-14-2\lambda\\6+5\lambda\end{pmatrix}$ $\overrightarrow{BN} \bullet \begin{pmatrix}0\\-2\\5\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}-2\\-14-2\lambda\\6+5\lambda\end{pmatrix} \bullet \begin{pmatrix}0\\-2\\5\end{pmatrix}$ $\Rightarrow 28+4\lambda+30+25\lambda=0$ $\Rightarrow 29\lambda=-58$ $\Rightarrow \lambda=-2$ $\overrightarrow{BN} = \begin{pmatrix}-2\\-10\\-4\end{pmatrix}$ Equation of line BN : $\underset{\sim}{r} = \begin{pmatrix}4\\13\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\5\\2\end{pmatrix}$

	Solution

5(i)

$$y=\cos|x-1|$$

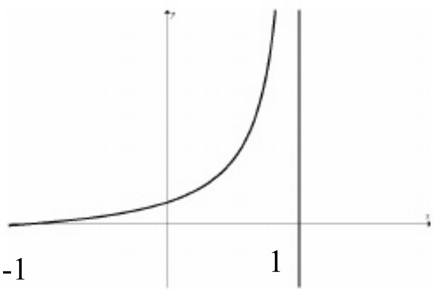
$$\cos^{-1} y=|x-1|$$

$$x=1\pm\cos^{-1} y$$

Since $1-\pi\le x\le 1$, $x=1-\cos^{-1} y$

$$f^{-1}: x\mapsto 1-\cos^{-1} x,\ x\in\mathbb{R},\ -1\le x\le 1$$

(ii)



$R_g=[0,\infty)$

$\not\subset D_f=[1-\pi,1]$

$\Rightarrow fg$ does not exist.

For fg to exist,

$R_g=[0,1]\Rightarrow$ maximal $D_g=[-1,0]$

6(i)

$$\int_0^1 \frac{x}{\sqrt{2-x}}\,dx=-\int_0^1 \frac{2-x-2}{\sqrt{2-x}}\,dx$$

$$=\int_0^1 [-(2-x)^{\frac{1}{2}}+2(2-x)^{-\frac{1}{2}}]\,dx$$

$$=\left[\frac{2}{3}(2-x)^{\frac{3}{2}}-4(2-x)^{\frac{1}{2}}\right]_0^1$$

$$=\frac{2}{3}(4\sqrt{2}-5)$$

(ii)

$$S=\frac{1}{n}\cdot\frac{\frac{1}{n}}{\sqrt{2-\frac{1}{n}}}+\frac{1}{n}\cdot\frac{\frac{2}{n}}{\sqrt{2-\frac{2}{n}}}+\ldots+\frac{1}{n}\cdot\frac{\frac{n}{n}}{\sqrt{2-\frac{n}{n}}}$$

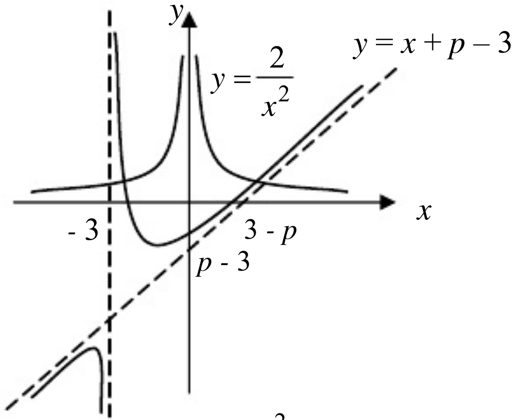
$$=\frac{1}{n}\left[\frac{\frac{1}{n}}{\sqrt{\frac{2n-1}{n}}}+\frac{\frac{2}{n}}{\sqrt{\frac{2n-2}{n}}}+\ldots+\frac{\frac{n}{n}}{\sqrt{\frac{2n-1}{n}}}\right]$$

$$=\frac{1}{n\sqrt{n}}\left[\frac{1}{\sqrt{2n-1}}+\frac{2}{\sqrt{2n-2}}+\ldots+\frac{n}{\sqrt{n}}\right]$$

$$\lim_{n\to\infty} S=\int_0^1 \frac{x}{\sqrt{2-x}}\,dx=\frac{2}{3}(4\sqrt{2}-5)$$

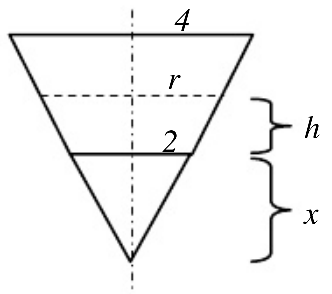
	Solution
	Alternative solution for (i) – by parts $\int_0^1 \frac{x}{\sqrt{2-x}}\,dx = \left[\left[-2x\sqrt{2-x} - \int -2\sqrt{2-x}\,dx\right]\right]_0^1$ $= \left[-2x\sqrt{2-x} + 2\left[-\frac{2}{3}(2-x)^{\frac{3}{2}}\right]\right]_0^1$ $= \left[-2x\sqrt{2-x} - \left[\frac{4}{3}(2-x)^{\frac{3}{2}}\right]\right]_0^1$ $= \frac{2}{3}(4\sqrt{2}-5)$
7a)	Area $\boldsymbol{A} = \int_1^2 \frac{\ln x}{x^2}\,dx - \frac{1}{2}\cdot 1\cdot\frac{\ln 2}{4}$ $= \left[(-\frac{1}{x})\ln x + \int \frac{1}{x}\cdot\frac{1}{x}\,dx\right]_1^2 - \frac{\ln 2}{8}$ $= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^2 - \frac{\ln 2}{8}$ $= \frac{1}{2} - \frac{5}{8}\ln 2$
7b)	Points of intersection of curves are $(-5, 9)$ and $(0, 4)$. Volume $= \pi\int_0^9 (-2-\sqrt{y})^2\,dy - \pi\int_0^4 (-2+\sqrt{y})^2\,dy - \pi\int_4^9 \left(\frac{16-y^2}{13}\right)^2 dy$ $= 466.52653 - 8.3775593 - 107.66306$ $= 350.4859107 \approx 350$
8i) ii)	After n leaps of the cheetah, the deer would have leaped $\frac{5}{6}n \times 2 = \frac{5}{3}n$. Therefore the deer is at a distance $\left(21\frac{2}{5} + \frac{5}{3}n\right)$ from the cheetah's starting point. Distance leaped by cheetah: $a = 4$, $d = -\frac{1}{10}$ After n leaps, the distance leaped by the cheetah = $S_c = \frac{n}{2}\left[8 - \frac{1}{10}(n-1)\right]$ To catch the deer, $S_c \geq 21\frac{2}{5} + \frac{5}{3}n$ $\frac{n}{2}\left[8 - \frac{1}{10}(n-1)\right] \geq 21\frac{2}{5} + \frac{5}{3}n$
	$4n - \frac{n}{20}(n-1) \geq \frac{107}{5} + \frac{5}{3}n$ $240n - 3n^2 + 3n \geq 1284 + 100n$ $3n^2 - 143n + 1284 \leq 0$ $(3n-107)(n-12) \leq 0 \Rightarrow 12 \leq n \leq 35\frac{2}{3}$ Least number of leaps =12

	Solution
iii)	Let k be the initial distance between the deer and the cheetah. For the deer to survive the chase, for all n values, $S_c < k+\frac{5}{3}n$ $\frac{n}{2}\left[8-\frac{1}{10}(n-1)\right] < k+\frac{5}{3}n$ $240n-3n^2+3n<60k+100n$ $3n^2-143n+60k>0$ $\Rightarrow$ Discriminant < 0 $\Rightarrow 143^2-720k<0$ $\Rightarrow \quad k>28.401$ m least distance = 28.5 m
9a)	let $z=x+yi$ $(x+yi+i)^*=2i((x+yi))+i$ $x-(y+1)i=-2y+i(2x+1)$ $\therefore$ $x=-2y$ ----- (1) $-(y+1)=2x+1$ ------(2) Sub (1) into (2): $-y-1=2(-2y)+1$ $\therefore y=\frac{2}{3}$ and $x=-\frac{4}{3}$
9b.	i) $z^5-1=0\Rightarrow z=e^{i\frac{2k\pi}{5}}$ $\Rightarrow z=e^{i\frac{2\pi}{5}},e^{-i\frac{2\pi}{5}},e^{i\frac{4\pi}{5}},e^{-i\frac{4\pi}{5}},1$ Accept e^{i0} ii) $(z+5)^5-(z-5)^5=0$ $\Rightarrow\left(\frac{5+z}{5-z}\right)^5=1$ $\Rightarrow\frac{5+z}{5-z}=e^{i\frac{2k\pi}{5}}$, k = 0, $\pm1,\pm2$ (from (i)) $\Rightarrow(5+z)=(5-z)e^{i\left(\frac{2k\pi}{5}\right)}$ $\Rightarrow z=\frac{5\left(e^{i\left(\frac{2k\pi}{5}\right)}-1\right)}{\left(e^{i\left(\frac{2k\pi}{5}\right)}+1\right)}$ $=\frac{5e^{i\left(\frac{k\pi}{5}\right)}\left(e^{i\left(\frac{k\pi}{5}\right)}-e^{i\left(\frac{-k\pi}{5}\right)}\right)}{e^{i\left(\frac{k\pi}{5}\right)}\left(e^{i\left(\frac{k\pi}{5}\right)}+e^{i\left(\frac{-k\pi}{5}\right)}\right)}$

	Solution
	$= \dfrac{5.2i\sin\left(\frac{k\pi}{5}\right)}{2\cos\left(\frac{k\pi}{5}\right)} = \quad 5i\tan\dfrac{k\pi}{5}$ (proved)
10i)	$y=\dfrac{x^2+px-q}{x+3}=x+p-3+\dfrac{9-q-3p}{x+3} \qquad \Rightarrow$ Asymptotes: $y=x+p-3$, $\quad x=-3$
ii)	$\dfrac{dy}{dx}=1-\dfrac{9-q-3p}{(x+3)^2}$ For $\dfrac{dy}{dx}=0$, $(x+3)^2=9-q-3p$ $x=-3\pm\sqrt{9-q-3p}$ For 2 turning points, $9-q-3p>0$ $\Rightarrow q<9-3p$ (shown) iii)  When $p=2$, $q=1$, $y=\dfrac{x^2+2x-1}{x+3}$ $x^4+2x^3-x^2-2x-6=0$ $x^2\left(x^2+2x-1\right)=2(x+3)$ $\dfrac{x^2+2x-1}{x+3}=\dfrac{2}{x^2}$ ----- (1) 2 intersection points between C & $y=\dfrac{2}{x^2} \Rightarrow$ 2 real roots (shown)
11a)	(i) $u_{r+2}=u_{r+1}+u_r$ $\dfrac{u_{r+2}}{u_{r+1}}=1+\dfrac{u_r}{u_{r+1}}$ $v_{r+1}=1+\dfrac{1}{v_r}$ (ii) As $r\to\infty$, $v_r\to k$ and $v_{r+1}\to k$ $\therefore \quad k=1+\dfrac{1}{k}$

	Solution
	$k^2 = k+1$ $k^2 - k - 1 = 0$ $k = \frac{1 \pm \sqrt{5}}{2}$ Since $u_r > 0$ for all $r \geq 1$ $\Rightarrow v_r = \frac{u_{r+1}}{u_r} > 0$ for all $r \geq 1$ $\Rightarrow k = \frac{1+\sqrt{5}}{2}$ (ans)
11b)	(i) $u_1 = 1$ $u_2 = 1$ $u_3 = u_1 + u_2 = 2$ $u_4 = u_1 + u_2 + u_3 = 4$ $u_5 = u_1 + u_2 + u_3 + u_4 = 8$ $u_6 = u_1 + u_2 + u_3 + u_4 + u_5 = 16$ (ii) $u_n = 2^{n-2}, \quad n \geq 2$ (iii) Let $n = 2$, LHS = $u_2 = 1$ RHS = $2^0 = 1$ Therefore the result is true for n = 2. Assume that the result is true for n = k, $k \geq 2$ i.e. $u_k = \sum_{i=1}^{k-1} u_i = 2^{k-2}, \quad k \geq 2$ For $n = k+1$, $u_{k+1} = \sum_{i=1}^{k} u_i = \sum_{i=1}^{k-1} u_i + u_k$ $= 2^{k-2} + 2^{k-2} = 2.2^{k-2}$ $= 2^{k-1} = 2^{(k+1)-2}$ Therefore the result is true for $n = k+1$. Hence by induction, the result is true for all $n \in Z$, $n \geq 2$.
12a)	a) $\frac{dy}{dx} + [1 + (x-y)^2] \cos^2 x = \sin^2 x$ ---------------------- (1) Using $v = x - y$, $\frac{dv}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$ --------------------- (2) Substitute (2) into (1) : $1 - \frac{dv}{dx} + [1 + v^2] \cos^2 x = \sin^2 x$ $\frac{dv}{dx} = [1 + v^2][\cos^2 x] + 1 - \sin^2 x$ $\frac{dv}{dx} = \cos^2 x [2 + v^2]$

	Solution
	$\int \frac{1}{2+v^2}\,dv = \frac{1}{2}\int(1+\cos 2x)\,dx$ $\frac{1}{\sqrt{2}}\tan^{-1}\frac{v}{\sqrt{2}} = \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right) + C$ $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-y}{\sqrt{2}} = \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right) + C$ $y = x - \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}\left(x+\frac{1}{2}\sin 2x\right)+\sqrt{2}C\right)$
12b)	$\frac{dx}{dt} = R - kx$, k is a positive constant At $x = 1.5\,R$, $\frac{dx}{dt} = 0 \quad \Rightarrow R - \frac{3}{2}Rk = 0 \Rightarrow \text{k} = \frac{2}{3}$ Thus , $\frac{dx}{dt} = R - \frac{2}{3}x$ (shown) i) $\int \frac{1}{R-\frac{2}{3}x}\,dx = \int 1\,dt$ $-\frac{1}{\cancel{2/3}}\ln\left\lvert R - \frac{2}{3}x\right\rvert = t + c$ $\ln\left\lvert R - \frac{2}{3}x\right\rvert = -\frac{2}{3}t - \frac{2}{3}c$ $R - \frac{2}{3}x = A\,e^{-\frac{2}{3}t}$ $x = \frac{3}{2}\left(R - A\,e^{-\frac{2}{3}t}\right)$ At $t = 0$, $x = 0$, $0 = (R - A)$ ie $A = R$ $\Rightarrow \quad x = \frac{3R}{2}\left(1 - e^{-\frac{2t}{3}}\right)$ ii) As $t \to \infty$, $e^{-\frac{2}{3}t} \to 0$ $\text{x} \to \frac{3}{2}R \Rightarrow \alpha = \frac{3}{2}R$ ie regardless of time, the amount of drug in the patient's body will never exceed $\frac{3}{2}R$.
13a)	$L = \sqrt{y^2 + (18-x)^2}$ $= \sqrt{x^4 + (18-x)^2}$ $L^2 = x^4 + (18-x)^2$ $2L\frac{dL}{dx} = 4x^3 + 2(18-x)(-1)$

	Solution
	At min pt, $\frac{dL}{dx}=0$ $\therefore 4x^3 = 36-2x$ $2x^3+x-18=0$ From GC, x =2 is the only solution. Therefore the point is (2, 4) $x=2^+, \frac{dL}{dx}>0$ $x=2^-, \frac{dL}{dx}<0 \quad \therefore$ Min point.
13b)	i)  $\frac{2}{4}=\frac{x}{6+x} \Rightarrow x=6$ $\frac{2}{r}=\frac{6}{6+h} \Rightarrow r=2+\frac{h}{3}$ $V=\frac{1}{3}\pi r^2(h+6)-\frac{1}{3}\pi(2^2)6$ $=\frac{1}{3}\pi\left(2+\frac{h}{3}\right)^2(h+6)-8\pi$ $=\frac{1}{3}\pi\left(\frac{1}{3}\right)^2(h+6)^3-8\pi$ $=\frac{\pi}{27}(h+6)^3-8\pi$ ii) $\frac{dV}{dh}=\frac{\pi}{9}(h+6)^2$ $\frac{dh}{dt}=\frac{dh}{dv}\times\frac{dv}{dt}$ $=\frac{9}{\pi(h+6)^2}(20) = \frac{180}{\pi(h+6)^2}$ $when\ h=3cm,$ $\frac{dh}{dt}=\frac{180}{\pi(81)} = \frac{20}{9\pi}\ cm/s$