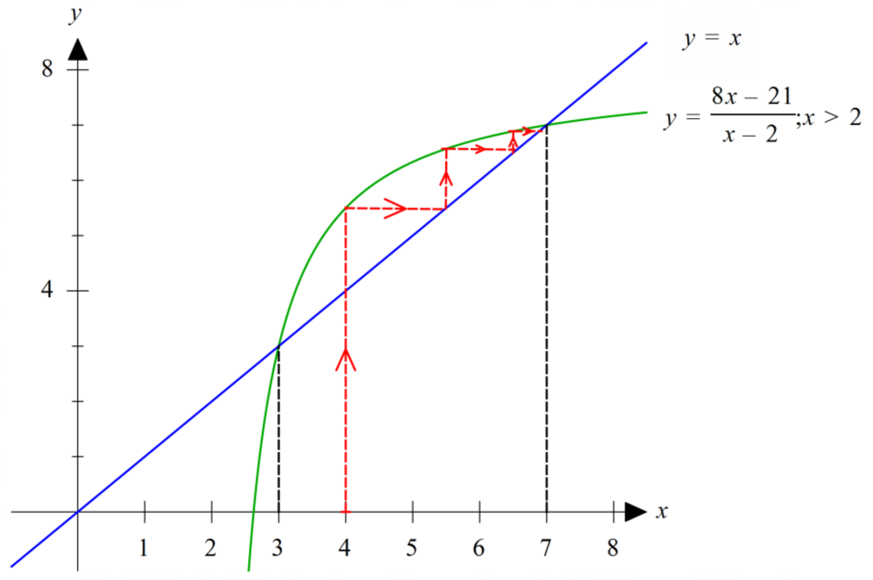
FM Prelim Paper 1 (100 marks)

1(a) **[2]**	Let $\mathrm{f}(x) = x^3 - \mathrm{e}^{-2x} - 2$ $\mathrm{f}(1) = (1)^3 - \mathrm{e}^{-2(1)} - 2 = -1.1353$ (5 s.f.) < 0 $\mathrm{f}(2) = (2)^3 - \mathrm{e}^{-2(2)} - 2 = 5.9817$ (5 s.f.) > 0 [Alternatively, since $0 < \mathrm{e}^{-2} < 1$ and $0 < \mathrm{e}^{-4} < 1$, $\mathrm{f}(1) = (1)^3 - \mathrm{e}^{-2(1)} - 2 = -1 - \mathrm{e}^{-2} < 0$ $\mathrm{f}(2) = (2)^3 - \mathrm{e}^{-2(2)} - 2 = 6 - \mathrm{e}^{-4} > 0$] Since $\mathrm{f}'(x) = 3x^2 + 2\mathrm{e}^{-2x} > 0 \quad \forall x \in \mathbb{R}$, f is strictly increasing and continuous. Hence there is exactly one root, α, that lies in [1,2].	
(b) **[2]**	Let $\mathrm{f}(x) = x^3 - \mathrm{e}^{-2x} - 2$ Using linear interpolation on [1,2], we have $\alpha_1 = \dfrac{(1)\mathrm{f}(2) - (2)\mathrm{f}(1)}{\mathrm{f}(2) - \mathrm{f}(1)} = 1.1595 = 1.2$ (1 d.p.)	
(c) **[2]**	$\mathrm{f}'(x) = 3x^2 + 2\mathrm{e}^{-2x}$ By Newton-Raphson method, we have $\alpha_{n+1} = \alpha_n - \dfrac{\mathrm{f}(\alpha_n)}{\mathrm{f}'(\alpha_n)} = \alpha_n - \dfrac{\alpha_n{}^3 - \mathrm{e}^{-2\alpha_n} - 2}{3\alpha_n{}^2 + 2\mathrm{e}^{-2\alpha_n}}$ Take the initial approximation to be $\alpha_1 = 1.2$, $\alpha_2 = 1.28058$, $\alpha_3 = 1.27609$, $\alpha_4 = 1.27607$ <u>Checking</u> $\mathrm{f}(1.2755) = -0.00289 < 0$ $\mathrm{f}(1.2765) = 0.00215 > 0$ That is, $\alpha = 1.276$ (3 d.p.).	

2(a) **[2]**	Sequence converges $\Rightarrow$ as $n \to \infty$, $u_n \to l$ and $u_{n+1} \to l$ $l = \dfrac{8l - 21}{l - 2}$ $\Rightarrow l$ satisfies the equation $\dfrac{8x - 21}{x - 2} = x$,	

	$\frac{8x-21}{x-2}=x$ $x(x-2)=8x-21$ $x^2-10x+21=0$ $x=3$ or $x=7$ Hence the sequence converges to either 3 or 7. (proven)	
(b) **[4]**	$u_{n+1}-7=\frac{8u_n-21}{u_n-2}-7$ $=\frac{8u_n-21-7(u_n-2)}{u_n-2}$ $=\frac{u_n-7}{u_n-2}<0$ $(\because u_n-7<0,\ u_n-2>0)$ $\Rightarrow u_{n+1}<7$ Next, consider $u_{n+1}-u_n$, $u_{n+1}-u_n=\frac{8u_n-21}{u_n-2}-u_n$ $=\frac{8u_n-21-u_n(u_n-2)}{u_n-2}$ $=-\frac{u_n^2-10u_n+21}{u_n-2}$ $=-\frac{(u_n-3)(u_n-7)}{u_n-2}$ Since $u_n>2$, $u_n-2>0$. Since $3<u_n<7$, $(u_n-3)(u_n-7)<0$. Hence $u_{n+1}-u_n=-\frac{(u_n-3)(u_n-7)}{u_n-2}>0$ $\Rightarrow u_{n+1}-u_n>0$ for $3<u_n<7$ Thus $u_{n+1}>u_n$ if $3<u_n<7$. Thus, $u_n<u_{n+1}<7$ if $3<u_n<7$.	

(c) **[2]**	 From the graph, the sequence strictly increases and converges to 7. Alternatively, Apply **(b)** repeatedly. $3 < u_1 < 7 \quad \Rightarrow u_1 < u_2 < 7$ $\Rightarrow 3 < u_2 < 7$ $\Rightarrow u_2 < u_3 < 7$ $\Rightarrow 3 < u_3 < 7$ $\Rightarrow u_3 < u_4 < 7 \quad \Rightarrow \ldots$ Hence we have $3 < u_1 < u_2 < u_3 < u_4 < u_5 < \ldots < u_n < u_{n+1} < \ldots < 7$, and so the sequence strictly increases and converges to 7.	

3(a) **[3]**	The characteristic equation for $2ax_{n+2} - 2\sqrt{a}\,x_{n+1} + x_n = 0$ is $2a\lambda^2 - 2\sqrt{a}\,\lambda + 1 = 0$, and so for $a > 0$, $\lambda = \dfrac{2\sqrt{a} \pm \sqrt{4a - 4(2a)(1)}}{2(2a)} = \dfrac{\sqrt{a} \pm \sqrt{a - 2a}}{2a}$ (2 distinct roots) $= \dfrac{\sqrt{a} \pm \sqrt{-a}}{2a} = \dfrac{\sqrt{a} \pm i\sqrt{a}}{2a} = \dfrac{1}{2\sqrt{a}}(1 \pm i) = \dfrac{\sqrt{2}}{2\sqrt{a}} e^{\pm i\frac{\pi}{4}}$ Hence the general solution is $x_n = A\left(\dfrac{1+i}{2\sqrt{a}}\right)^n + B\left(\dfrac{1-i}{2\sqrt{a}}\right)^n$.	

	or $x_n=\left(\frac{\sqrt{2}}{2\sqrt{a}}\right)^n\left((A+B)\cos\left(\frac{n\pi}{4}\right)+(A-B)i\sin\left(\frac{n\pi}{4}\right)\right)$ (note that A and B are complex numbers) or $x_n=\left(\frac{\sqrt{2}}{2\sqrt{a}}\right)^n\left(C\cos\left(\frac{n\pi}{4}\right)+D\sin\left(\frac{n\pi}{4}\right)\right),\ C,D\in\mathbb{R}$	
(b) **[3]**	When $a=3$, the general solution is $x_n=\left(\frac{\sqrt{2}}{2\sqrt{3}}\right)^n\left(C\cos\left(\frac{n\pi}{4}\right)+D\sin\left(\frac{n\pi}{4}\right)\right),\ C,D\in\mathbb{R}$ $x_0=3\Rightarrow C=3$ $x_1=\frac{1}{\sqrt{3}}\Rightarrow\frac{\sqrt{2}}{2\sqrt{3}}\left(\frac{3}{\sqrt{2}}+D\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{3}}\Rightarrow 3+D=2\Rightarrow D=-1$ Hence $x_n=\left(\frac{1}{\sqrt{6}}\right)^n\left(3\cos\left(\frac{n\pi}{4}\right)-\sin\left(\frac{n\pi}{4}\right)\right)$, and so $C=3,\ D=-1,\ \mathrm{f}(n)=\left(\frac{1}{\sqrt{6}}\right)^n$, $\mathrm{g}(n)=\frac{n\pi}{4}$. Alternatively When $a=3$, the general solution is $x_n=A\left(\frac{1+i}{2\sqrt{3}}\right)^n+B\left(\frac{1-i}{2\sqrt{3}}\right)^n$ $x_0=3\Rightarrow A+B=3$ $x_1=\frac{A}{2\sqrt{3}}(1+i)+\frac{B}{2\sqrt{3}}(1-i)=\frac{1}{2\sqrt{3}}\left(A+B+(A-B)i\right)$ $\therefore\frac{1}{2\sqrt{3}}\left(3+(A-B)i\right)=\frac{1}{\sqrt{3}}\Rightarrow 3+(A-B)i=2\Rightarrow A-B=-\frac{1}{i}=i$ $\therefore A=\frac{1}{2}(3+i),\ B=\frac{1}{2}(3-i)$ Hence	

	$x_n = \frac{3+i}{2}\left(\frac{1+i}{2\sqrt{3}}\right)^n + \frac{3-i}{2}\left(\frac{1-i}{2\sqrt{3}}\right)^n$ $= \frac{3+i}{2}\left(\frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2\sqrt{3}}\right)^n + \frac{3-i}{2}\left(\frac{\sqrt{2}e^{-i\frac{\pi}{4}}}{2\sqrt{3}}\right)^n$ $= \frac{3+i}{2(\sqrt{6})^n}\left(\cos\left(\frac{n\pi}{4}\right)+i\sin\left(\frac{n\pi}{4}\right)\right)$ $+ \frac{3-i}{2(\sqrt{6})^n}\left(\cos\left(\frac{n\pi}{4}\right)-i\sin\left(\frac{n\pi}{4}\right)\right)$ $= \frac{1}{2(\sqrt{6})^n}\left(6\cos\left(\frac{n\pi}{4}\right)-2\sin\left(\frac{n\pi}{4}\right)\right)$ $= \frac{1}{(\sqrt{6})^n}\left(3\cos\left(\frac{n\pi}{4}\right)-\sin\left(\frac{n\pi}{4}\right)\right)$	
(c) **[2]**	For $a > 0$, The general solution is $x_n = \left(\frac{\sqrt{2}}{2\sqrt{a}}\right)^n \left(C\cos\left(\frac{n\pi}{4}\right)+D\sin\left(\frac{n\pi}{4}\right)\right),\ C, D \in \mathbb{R}$ Note that the trigonometric terms are periodic and bounded. Hence for $x_n \to 0$, we only require $0 < \frac{\sqrt{2}}{2\sqrt{a}} < 1 \Rightarrow \sqrt{a} > \frac{1}{\sqrt{2}} \Rightarrow a > \frac{1}{2} \Rightarrow \left(\frac{1}{2}, \infty\right)$ is the required range.	

4(a) **[3]**	Since $\lvert\lambda_1\rvert > \lvert\lambda_2\rvert > \lvert\lambda_3\rvert > \lvert\lambda_4\rvert > 0$, the eigenvalues are distinct and so $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is linearly independent. $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ linearly independent → $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a basis of $\mathbb{R}^4$. Hence for any $\mathbf{x} \in \mathbb{R}^4$, it can be expressed as a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4 \Rightarrow \mathbf{Ax} = \mathbf{A}c_1\mathbf{x}_1 + \mathbf{A}c_2\mathbf{x}_2 + \mathbf{A}c_3\mathbf{x}_3 + \mathbf{A}c_4\mathbf{x}_4$ $= c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + c_3\mathbf{Ax}_3 + c_4\mathbf{Ax}_4$ $= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + c_3\lambda_3\mathbf{x}_3 + c_4\lambda_4\mathbf{x}_4$ By pre-multiplying by $\mathbf{A}$ repeatedly, we have	

	$\mathbf{A}^2\mathbf{x} = \mathbf{A}c_1\lambda_1\mathbf{x}_1 + \mathbf{A}c_2\lambda_2\mathbf{x}_2 + \mathbf{A}c_3\lambda_3\mathbf{x}_3 + \mathbf{A}c_4\lambda_4\mathbf{x}_4$ $= c_1\lambda_1\mathbf{A}\mathbf{x}_1 + c_2\lambda_2\mathbf{A}\mathbf{x}_2 + c_3\lambda_3\mathbf{A}\mathbf{x}_3 + c_4\lambda_4\mathbf{A}\mathbf{x}_4$ $= c_1\lambda_1^2\mathbf{x}_1 + c_2\lambda_2^2\mathbf{x}_2 + c_3\lambda_3^2\mathbf{x}_3 + c_4\lambda_4^2\mathbf{x}_4$ $\mathbf{A}^3\mathbf{x} = c_1\lambda_1^3\mathbf{x}_1 + c_2\lambda_2^3\mathbf{x}_2 + c_3\lambda_3^3\mathbf{x}_3 + c_4\lambda_4^3\mathbf{x}_4$ $\mathbf{A}^k\mathbf{x} = c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + c_3\lambda_3^k\mathbf{x}_3 + c_4\lambda_4^k\mathbf{x}_4$ By factorizing out λ_1^k, we have $\mathbf{A}^k\mathbf{x} = \lambda_1^k\left(c_1\mathbf{x}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + c_3\left(\frac{\lambda_3}{\lambda_1}\right)^k\mathbf{x}_3 + c_4\left(\frac{\lambda_4}{\lambda_1}\right)^k\mathbf{x}_4\right)$	
(b) [1]	From (a), since $\lvert\lambda_1\rvert > \lvert\lambda_2\rvert > \lvert\lambda_3\rvert > \lvert\lambda_4\rvert > 0,$ we observe that for large k, $\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0$, $i = 2,3,4$, since $\left\lvert\frac{\lambda_i}{\lambda_1}\right\rvert < 1$, and so $\mathbf{A}^k\mathbf{x} = \lambda_1^k\left(c_1\mathbf{x}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + c_3\left(\frac{\lambda_3}{\lambda_1}\right)^k\mathbf{x}_3 + c_4\left(\frac{\lambda_4}{\lambda_1}\right)^k\mathbf{x}_4\right) \approx \lambda_1^k c_1\mathbf{x}_1$.	
(c) [1]	Using the GC to solve for **Mx** = –**x** as a SLE, we get the eigenvector associated with eigenvalue -1 as $\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$.	
(d) [3]	$\lambda_1 = 1.80194,\ \lambda_2 = -1.24698,\ \lambda_3 = -1,\ \lambda_4 = 0.44504,$ Note that the eigenvalues of **M** are all distinct, so results in (a) and (b) holds. Note that it suffices to find the least integer value of *k* by considering $\left\lvert\left(\frac{\lambda_2}{\lambda_1}\right)^k\right\rvert < 0.0001$ as $\left\lvert\left(\frac{\lambda_2}{\lambda_1}\right)^k\right\rvert > \left\lvert\left(\frac{\lambda_3}{\lambda_1}\right)^k\right\rvert > \left\lvert\left(\frac{\lambda_4}{\lambda_1}\right)^k\right\rvert$.From GC, <table><tr><td>k</td><td>$\left\lvert\left(\frac{\lambda_2}{\lambda_1}\right)^k\right\rvert < 0.0001$</td></tr><tr><td>25</td><td>0.0001</td></tr><tr><td>26</td><td>0.00007 < 0.0001</td></tr></table>Thus least *k* = 26	

	Since $\alpha^{26}c_1\mathbf{x}_1 \approx \mathbf{M}^{26}\mathbf{x} = \begin{pmatrix}1557649\\1942071\\1942071\\864201\end{pmatrix}$ and $\alpha^{26}c_1$ is a constant, $\mathbf{x}_1$ can be scaled accordingly, and so an approximate eigenvector $\mathbf{x}_1$ is $\begin{pmatrix}0.80\\1\\1\\0.44\end{pmatrix}$.	

5(a) **[3]**	$\det(\mathbf{A}-\lambda\mathbf{I}) =$ $(a-\lambda)(-a-\lambda)-(b+a)(b-a) = -(a^2-\lambda^2)-(b^2-a^2) = \lambda^2-b^2$. Hence $\det(\mathbf{A}-\lambda\mathbf{I}) = 0 \Rightarrow \lambda^2-b^2 = 0 \Rightarrow \lambda = \pm b$ Eigenvalues of **A** satisfy the characteristic equation $\lambda^2-b^2=0$ which has roots $\pm b$. Hence b is an eigenvalue of **A**. [Alternatively, Since the column sums of **A** are both b, $\mathbf{A}^{\mathrm{T}}\begin{pmatrix}1\\1\end{pmatrix} = b\begin{pmatrix}1\\1\end{pmatrix}$, and hence b is an eigenvalue of $\mathbf{A}^{\mathrm{T}}$ with eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$. But eigenvalue of **A** is same as eigenvector of $\mathbf{A}^{T}$. Hence b is an eigenvalue of **A.**] To find corresponding eigenvector, solve: $\begin{pmatrix}a-b & b+a\\b-a & -a-b\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ Since $\begin{pmatrix}a-b & b+a\\b-a & -a-b\end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix}a-b & b+a\\0 & 0\end{pmatrix}$, $y = \dfrac{b-a}{b+a}x$, and so an eigenvector corresponding to b is $\begin{pmatrix}1\\\dfrac{b-a}{b+a}\end{pmatrix}$ or $\begin{pmatrix}b+a\\b-a\end{pmatrix}$. [Alternatively, $(a-b \quad b+a)\begin{pmatrix}x\\y\end{pmatrix} = 0$, and so an eigenvector corresponding to b is perpendicular to $\begin{pmatrix}a-b\\b+a\end{pmatrix}$ which is $\begin{pmatrix}b+a\\b-a\end{pmatrix}$.]	

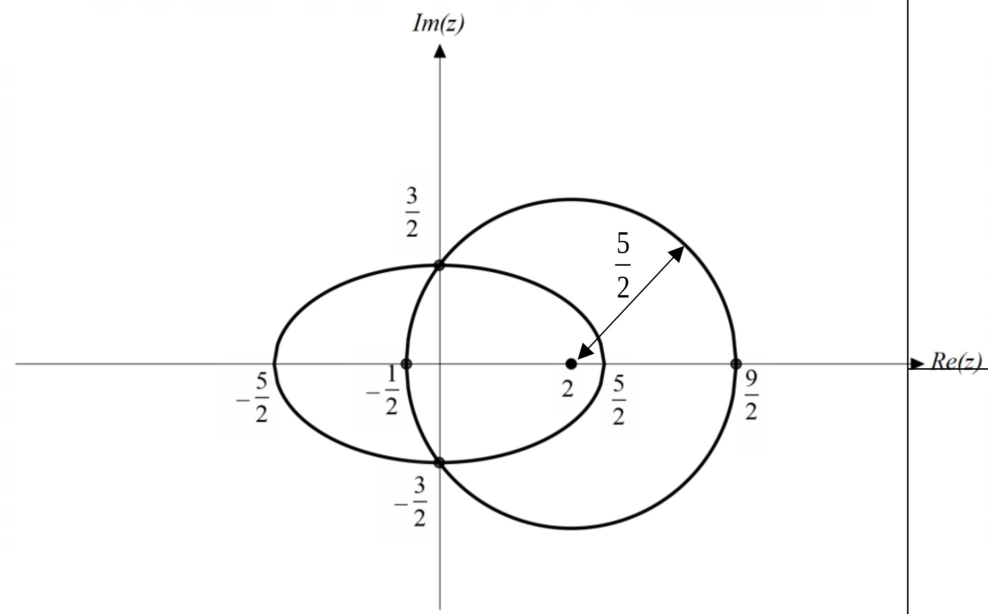
(b) **[3]**	$\mathbf{A}^2 = \begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix}\begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix}$ $= \begin{pmatrix} a^2+b^2-a^2 & 0 \\ 0 & b^2-a^2+a^2 \end{pmatrix} = \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix}$ $\therefore \mathbf{A}^2 = b^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = b^2\mathbf{I}_2$ $\mathbf{A}^2 = b^2\mathbf{I}_2$ $\mathbf{A}\left(\frac{1}{b^2}\mathbf{A}\right) = \mathbf{I}_2$ $\mathbf{A}^{-1} = \begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix}^{-1} = \frac{1}{b^2}\begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix}$ OR directly $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} -a & -(b+a) \\ -(b-a) & a \end{pmatrix}$ $= \frac{1}{-a^2-(b+a)(b-a)}\begin{pmatrix} -a & -(b+a) \\ -(b-a) & a \end{pmatrix}$ $= \frac{1}{-a^2-(b^2-a^2)}\begin{pmatrix} -a & -(b+a) \\ -(b-a) & a \end{pmatrix}$ $= \frac{1}{b^2}\begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix}$	
(c) **[5]**	$\mathbf{A} = \begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix} \Rightarrow \mathbf{A}^2 = \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix} = b^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = b^2\mathbf{I}_2$ $\Rightarrow \mathbf{A}^3 = b^2\begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix} = b^{3-1}\mathbf{A}$ $\Rightarrow \mathbf{A}^4 = b^2\mathbf{A}^2 = b^4\mathbf{I}_2$ $\Rightarrow \mathbf{A}^5 = b^4\mathbf{A} = b^{5-1}\mathbf{A}$ Hence $\mathbf{A}^n = \begin{cases} b^{n-1}\mathbf{A} & \text{if } n \text{ is odd} \\ b^n\mathbf{I}_2 & \text{if } n \text{ is even} \end{cases}$ Let P_n, where $n \geq 0$, be the statement $\mathbf{A}^n = \begin{cases} b^{n-1}\mathbf{A} & \text{if } n \text{ is odd} \\ b^n\mathbf{I}_2 & \text{if } n \text{ is even} \end{cases}$. Since $\mathbf{A} = b^{1-1}\mathbf{A}$, $\mathbf{A}^2 = b^2\mathbf{I}$, P_n is true for $n = 1, 2$.	

	Assume that P_k is true for **some** $k \in \mathbb{Z}^+$, that is $\mathbf{A}^k = \begin{cases} b^{k-1}\,\mathbf{A} & \text{if } k \text{ is odd} \\ b^k\,\mathbf{I}_2 & \text{if } k \text{ is even} \end{cases}$ Then $\mathbf{A}^{k+1} = \begin{cases} b^{k-1}\,\mathbf{AA} = b^{k-1}b^2\,\mathbf{I}_2 = b^{k+1}\,\mathbf{I}_2 \\ \qquad \text{if } k \text{ is odd, that is } k+1 \text{ is even} \\ b^k\,\mathbf{A} = b^{[(k+1)-1]}\,\mathbf{A} \\ \qquad \text{if } k \text{ is even, that is } k+1 \text{ is odd} \end{cases}$ Hence, P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 and P_2 is true, by Mathematical Induction, P_n is true for **all** $n \in \mathbb{Z}^+$.	

6(a) **[2]**	$\frac{1}{r} = A\cos\theta + B\cos(\theta-\alpha)$ $1 = Ar\cos\theta + Br(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$ $1 = Ax + B(x\cos\alpha + y\sin\alpha)$ $(A + B\cos\alpha)x + B\sin\alpha\, y = 1$ $mx + ny = 1$, where $m = A + B\cos\alpha$ and $n = B\sin\alpha$	
(b) **(i)** **[4]**	Let the equation of the chord PQ be $\frac{1}{r} = A\cos\theta + B\cos(\theta-\alpha)$ At P, $r_P = \frac{a}{1+e\cos(\alpha-\beta)}$ $\frac{1}{r_P} = \frac{1}{a}(1+e\cos(\alpha-\beta)) = A\cos(\alpha-\beta) + B\cos(-\beta)$ So, $\frac{1}{a}(1+e\cos(\alpha-\beta)) = A\cos(\alpha-\beta) + B\cos\beta \;\; --- \;\; (1)$ At Q, $r_Q = \frac{a}{1+e\cos(\alpha+\beta)}$ $\frac{1}{r_Q} = \frac{1}{a}(1+e\cos(\alpha+\beta)) = A\cos(\alpha+\beta) + B\cos(\beta)$ So, $\frac{1}{a}(1+e\cos(\alpha+\beta)) = A\cos(\alpha+\beta) + B\cos\beta \;\; --- \; (2)$ (2) − (1): $\frac{e}{a}(\cos(\alpha+\beta) - \cos(\alpha-\beta)) = A(\cos(\alpha+\beta) - \cos(\alpha-\beta))$	

	Comparing coefficients, $A=\frac{e}{a}$. Subst $A=\frac{e}{a}$ into (1), we get $\frac{1}{a}\left(1+e\cos(\alpha-\beta)\right)=\frac{e}{a}\cos(\alpha-\beta)+B\cos\beta$ $\frac{1}{a}=B\cos\beta$ $B=\frac{1}{a\cos\beta} \qquad 0<\beta<\frac{\pi}{2}\Rightarrow\cos\beta\neq 0$ Thus, $\frac{1}{r}=\frac{e}{a}\cos\theta+\frac{1}{a\cos\beta}\cos(\theta-\alpha)$ The equation of the chord PQ is $\frac{1}{r}=\frac{e}{a}\cos\theta+\frac{\sec\beta}{a}\cos(\theta-\alpha)$	
(ii) **[1]**	For the tangent at $\theta=\alpha$, let $\beta\to 0$. Then, $\sec\beta\to 1$. The equation of the tangent is $\frac{1}{r}=\frac{e}{a}\cos\theta+\frac{1}{a}\cos(\theta-\alpha)$.	

7(a) **[2+3]**	$z=w+\frac{1}{w}=k(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{k}(\cos\theta-\mathrm{i}\sin\theta)$ $=\left(k\cos\theta+\frac{1}{k}\cos\theta\right)+\mathrm{i}\left(k\sin\theta-\frac{1}{k}\sin\theta\right)$ $\mathrm{Re}(z)=x=\left(\frac{k^2+1}{k}\right)\cos\theta\Rightarrow\frac{k^2x^2}{(k^2+1)^2}=\cos^2\theta \quad ----(1)$ $\mathrm{Im}(z)=y=\left(\frac{k^2-1}{k}\right)\sin\theta\Rightarrow\frac{k^2y^2}{(k^2-1)^2}=\sin^2\theta \quad ----(2)$ $(1)+(2)\Rightarrow\frac{k^2x^2}{(k^2+1)^2}+\frac{k^2y^2}{(k^2-1)^2}=1$ $\Rightarrow\frac{x^2}{(k^2+1)^2}+\frac{y^2}{(k^2-1)^2}=\frac{1}{k^2}$ $\Rightarrow\frac{x^2}{\left(\frac{k^2+1}{k}\right)^2}+\frac{y^2}{\left(\frac{k^2-1}{k}\right)^2}=1$ Therefore z lies on an ellipse. Eccentricity = $\frac{2k}{k^2+1}$. [Note that the coordinates of the foci are $(-2,\ 0)$ and $(2,\ 0)$.]	

(b) **(i)** **(ii)** **[2+2]**	From **(a)** $z=w+\frac{1}{w}\Rightarrow\frac{x^2}{\left(\frac{2^2+1}{2}\right)^2}+\frac{y^2}{\left(\frac{2^2-1}{2}\right)^2}=1\Rightarrow\frac{x^2}{\left(\frac{5}{2}\right)^2}+\frac{y^2}{\left(\frac{3}{2}\right)^2}=1$ 	
(iii) **[2]**	$\arg\left(v-\frac{3}{2}\right)=\frac{3\pi}{4}$ or $-\frac{3\pi}{4}$	
(iv) **[1]**	$\|v+2\|+\|v-2\|=\sqrt{2^2+\left(\frac{3}{2}\right)^2}+\sqrt{(-2)^2+\left(\frac{3}{2}\right)^2}=\sqrt{\frac{25}{4}}+\sqrt{\frac{25}{4}}=5$	

8(a)
[4]

Method 1

$$\begin{pmatrix} a & -1 & -1 \\ 1 & b & 3 \\ 0 & 1 & c \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & b & 3 \\ a & -1 & -1 \\ 0 & 1 & c \end{pmatrix}$$

$$\xrightarrow{R_2 - aR_1} \begin{pmatrix} 1 & b & 3 \\ 0 & -1-ab & -1-3a \\ 0 & 1 & c \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & b & 3 \\ 0 & 1 & c \\ 0 & -1-ab & -1-3a \end{pmatrix}$$

$$\xrightarrow{R_3 + (1+ab)R_2} \begin{pmatrix} 1 & b & 3 \\ 0 & 1 & c \\ 0 & 0 & -1-3a+(1+ab)c \end{pmatrix}$$

Since det($\mathbf{A}^{\mathrm{T}}\mathbf{A}$) = det($\mathbf{A}^{\mathrm{T}}$) det($\mathbf{A}$) = [det($\mathbf{A}$)]^2 = 0 → det($\mathbf{A}$) = 0, and so rank (**A**) is not 3. From the REF, there are 2 leading 1's and so rank(**A**) = 2. By the rank-nullity theorem, nullity (**A**) = 1, and so dim(nullspace) = 1.

In addition, $-1-3a+(1+ab)c = 0 \Rightarrow c = \dfrac{1+3a}{1+ab}$.

Method 2

For $a \neq 0$:

$$\begin{pmatrix} a & -1 & -1 \\ 1 & b & 3 \\ 0 & 1 & c \end{pmatrix} \xrightarrow{R_2 - \frac{1}{a}R_1} \begin{pmatrix} a & -1 & -1 \\ 0 & b+\frac{1}{a} & 3+\frac{1}{a} \\ 0 & 1 & c \end{pmatrix}$$

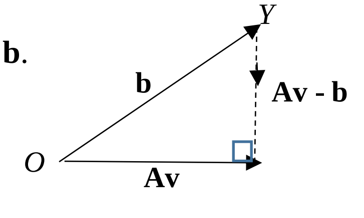
$$\xrightarrow{R_2 - (b+\frac{1}{a})R_3} \begin{pmatrix} a & -1 & -1 \\ 0 & 0 & 3+\frac{1}{a}-c(b+\frac{1}{a}) \\ 0 & 1 & c \end{pmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{-1}{a} & \frac{-1}{a} \\ 0 & 1 & c \\ 0 & 0 & 3+\frac{1}{a}-c(b+\frac{1}{a}) \end{pmatrix}$$

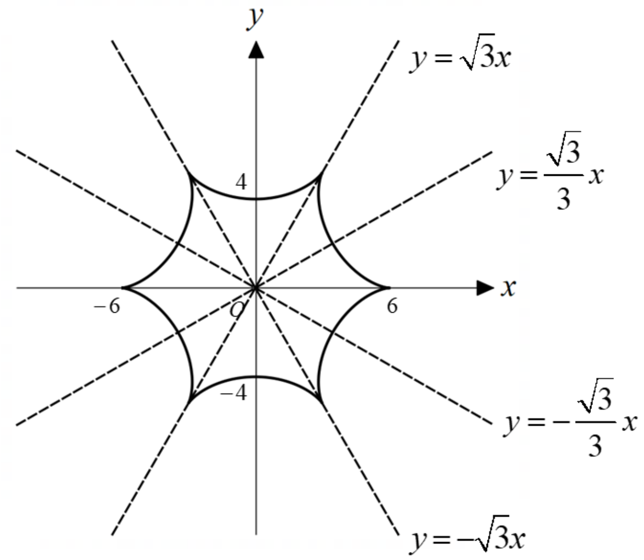
Since det($\mathbf{A}^{\mathrm{T}}\mathbf{A}$) = det($\mathbf{A}^{\mathrm{T}}$) det($\mathbf{A}$) = [det($\mathbf{A}$)]^2 = 0 → det($\mathbf{A}$) = 0, and so rank (**A**) is not 3.
From the REF, there are 2 leading 1's and so rank(**A**) = 2. By the rank-nullity theorem, nullity (**A**) = 1, and so dim(nullspace) = 1.

	Hence $3+\frac{1}{a}-c(b+\frac{1}{a})=0 \Rightarrow c=\dfrac{\frac{1}{a}(3a+1)}{\frac{1}{a}(ab+1)}=\dfrac{1+3a}{1+ab}$. For $a=0$, $\mathbf{A}=\begin{pmatrix}0 & -1 & -1\\ 1 & b & 3\\ 0 & 1 & c\end{pmatrix}$ and since det($\mathbf{A}$) $=-\left[-c-(-1)\right]=$ 0, $c=1$, which satisfies the equation also $\left(\because \dfrac{1+3(0)}{1+b(0)}=1\right)$. Alternative method to show c: det **(A) =** 0 $\Rightarrow abc-1-(-c+3a)=0 \Rightarrow c(ab+1)=1+3a \Rightarrow c=\dfrac{1+3a}{1+ab}$	
(b) **[1]**	Consider $\mathbf{A}=\begin{pmatrix}1 & -1 & -1\\ 1 & 0 & 3\\ 0 & 1 & 4\end{pmatrix}$. Since rank (**A**) = 2 and none of the columns are multiples of each other, any 2 of the columns is a basis of the column space of **A**. Geometrically, Column Space (**A**) is a plane passing through the origin and parallel to $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$. (or any other pair of columns in **A**) Alternatively : Plane containing origin and perpendicular to $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ Note : equation of the plane is $x-y+z=0$	
(c) **[1]**	Method 1 : consider subspace Since the column space of **A** is a plane through the origin, it is a subspace. Hence **Ax** is an element in this subspace. For **Ax =** $\begin{pmatrix}1\\0\\1\end{pmatrix}$ to have no solution, $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is NOT an element in the subspace**.** Hence, as subspaces are closed under scalar multiplication,	

	any multiple of $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is also NOT an element of the subspace. So $\begin{pmatrix}2\\0\\2\end{pmatrix}$ is not in the column space of **A**. Method 2 : proof by contradiction If $\begin{pmatrix}2\\0\\2\end{pmatrix}$ belong to the column space of **A**, then there exits **u** such that $\mathbf{Au} = \begin{pmatrix}2\\0\\2\end{pmatrix}$. Then $\mathbf{x} = \frac{1}{2}\mathbf{u}$ is a solution for $\mathbf{Ax} = \begin{pmatrix}1\\0\\1\end{pmatrix}$, a contradiction. Method 3 : by considering the geometrical relation between vector and plane For $\mathbf{Ax} = \begin{pmatrix}1\\0\\1\end{pmatrix}$ to have no solution, $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is not in the column space of **A**, and so it is not on the plane (from (b)). Since the plane (from (b)) contains the origin, $k\begin{pmatrix}1\\0\\1\end{pmatrix}$ is also not on the plane for all k (as it is a vector in the same direction), and so it is not in the column space of **A** also.	
(d) **[3]**	$\mathbf{A} = \begin{pmatrix}1 & -1 & -1\\1 & 0 & 3\\0 & 1 & 4\end{pmatrix}$ Least squares solutions to $\mathbf{Ax} = \begin{pmatrix}1\\0\\1\end{pmatrix}$ is **v** such that **v** is a solution to $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\begin{pmatrix}1\\0\\1\end{pmatrix}$. $\mathbf{A}^T\mathbf{Av} = \mathbf{A}^T\begin{pmatrix}1\\0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}2 & -1 & 2\\-1 & 2 & 5\\2 & 5 & 26\end{pmatrix}\mathbf{v} = \begin{pmatrix}1\\0\\3\end{pmatrix}$	

	Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then we have $\begin{aligned} 2x - y + 2z &= 1 \\ -x + 2y + 5z &= 0 \\ 2x + 5y + 26 &= 3 \end{aligned}$ From GC, $\mathbf{v} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$.	
(e) **[3]**	$\mathbf{Av} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix} \left[\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ $\mathbf{Av} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$ $\left(\mathbf{Av} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \bullet \mathbf{Av} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = -\frac{2}{9} + \frac{4}{9} - \frac{2}{9} = 0$, and so they are perpendicular. Let $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ be $\mathbf{b}$.  Shortest distance from Y to plane (column space of A) Note : **Av** is the projection of **b** onto the plane spanned by the columns of **A**.	

(f) **[2]**	$W=\{(x,y,z)\mid x+z=y,\ x,y,z\in\mathbb{R}\}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=y\begin{pmatrix}1\\1\\0\end{pmatrix}+z\begin{pmatrix}-1\\0\\1\end{pmatrix}$. Hence W = plane spanned by the columns of **A**. [Alternatively, W is the set containing the points on the plane $x-y+z=0$] Using (e), we identified that **Av** is the projection of $\begin{pmatrix}1\\0\\1\end{pmatrix}$ onto the plane spanned by the columns of **A.** This problem translates to finding a least squares solution, **v**, of $\mathbf{Ax}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ and then finding **Av**. From (e), $\mathbf{Av}=\begin{pmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$ Hence the nearest point is $\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)$.	

9(a) **[4]**	 Equations of the rest of the lines of symmetry :	

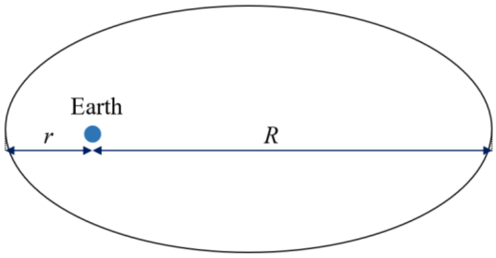
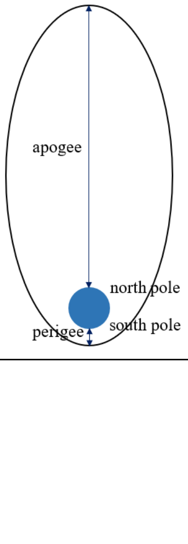
	$y=0, y=-\dfrac{\sqrt{3}}{3}x,\ y=-\sqrt{3}x,\ x=0$	
(b) **(i)** **[1]**	$x=r\cos\alpha=5\cos\theta+\cos5\theta,$ $y=r\sin\alpha=5\sin\theta-\sin5\theta.$ Hence it is clear that $\theta\neq\alpha$, and so the integral doesn't give the area. Alternatively, Area of A = $\int_{\alpha_1}^{\alpha_2}\dfrac{1}{2}r^2\ \mathrm{d}\alpha$ where α_1, α_2 correspond to the polar angle that S and T makes with the polar axis. However, in the integral $\int_0^{\tan^{-1}\frac{3}{2}}\dfrac{1}{2}\left((5\cos\theta+\cos5\theta)^2+(5\sin\theta-\sin5\theta)^2\right)\mathrm{d}\theta,$ θ is the parameter, and not the polar angle α. In particular, • $\theta=\tan^{-1}\dfrac{3}{2}$ does not correspond to T, and • $r^2\neq(5\cos\theta+\cos5\theta)^2+(5\sin\theta-\sin5\theta)^2$ Hence the area cannot be determined by the integral.	
b(ii) **[3]**	$\alpha_1=0\Rightarrow\dfrac{y}{x}=0\Rightarrow y=0\Rightarrow\theta=0$ $\alpha_2=\tan^{-1}\dfrac{3}{2}\Rightarrow\dfrac{y}{x}=\dfrac{3}{2}$ $\Rightarrow\dfrac{5\sin\theta-\sin5\theta}{5\cos\theta+\cos5\theta}=\dfrac{3}{2}$ $\Rightarrow\theta=0.7853982$ (from GC ... note that actual value is $\dfrac{\pi}{4}$) $x=5\cos\theta+\cos5\theta\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=-5(\sin\theta+\sin5\theta)$ $y=5\sin\theta-\sin5\theta\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=5(\cos\theta-\cos5\theta)$ Area of A $=\dfrac{1}{2}\int_0^{0.7853982}\begin{bmatrix}(5\cos\theta+\cos5\theta)(5(\cos\theta-\cos5\theta))\\-(5\sin\theta-\sin5\theta)(-5(\sin\theta+\sin5\theta))\end{bmatrix}\mathrm{d}\theta$ $=9.520648667$ (Using GC to integrate) $=9.52$ units^2 (3 s.f.)	
(c) **[5]**	$x=k\left(5\cos\theta+\cos5\theta\right)\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=-5k\left(\sin\theta+\sin5\theta\right)$ $y=k\left(5\sin\theta-\sin5\theta\right)\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=5k\left(\cos\theta-\cos5\theta\right)$	

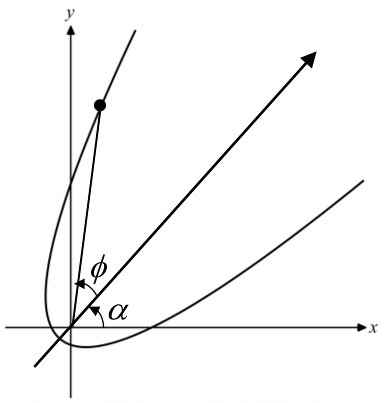
$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}$$
$$=\sqrt{\left(-5k\left(\sin\theta+\sin 5\theta\right)\right)^2+\left(5k\left(\cos\theta-\cos 5\theta\right)\right)^2}$$
$$=5k\sqrt{\left(\sin\theta+\sin 5\right)^2+\left(\cos\theta-\cos 5\theta\right)^2}$$
$$=5k\sqrt{\sin^2\theta+\sin^2 5\theta+2\sin\theta\sin 5\theta+\cos^2\theta+\cos^2 5\theta-2\cos\theta\cos 5\theta}$$
$$=5k\sqrt{2\sin\theta\sin 5\theta-2\cos\theta\cos 5\theta+2}$$
$$=5k\sqrt{2-2\cos 6\theta}$$
$$=5k\sqrt{4\sin^2 3\theta}$$
$$=10k\left|\sin 3\theta\right|$$

Required Surface area

$$=2\times\int_0^{\frac{\pi}{2}}2\pi x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$
$$=2\times\int_0^{\frac{\pi}{2}}2\pi k\left(5\cos\theta+\cos 5\theta\right)10k\left|\sin 3\theta\right|\,\mathrm{d}\theta$$
$$=40\pi k^2\int_0^{\frac{\pi}{3}}\left(5\cos\theta+\cos 5\theta\right)\left[\sin 3\theta\right]\,\mathrm{d}\theta$$
$$-40\pi k^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(5\cos\theta+\cos 5\theta\right)\left[\sin 3\theta\right]\,\mathrm{d}\theta$$
$$=20\pi k^2\int_0^{\frac{\pi}{3}}5\sin 4\theta+5\sin 2\theta+\sin 8\theta-\sin 2\theta\;\mathrm{d}\theta$$
$$-20\pi k^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}5\sin 4\theta+5\sin 2\theta+\sin 8\theta-\sin 2\theta\;\mathrm{d}\theta$$
$$=20\pi k^2\int_0^{\frac{\pi}{3}}5\sin 4\theta+4\sin 2\theta+\sin 8\theta\;\mathrm{d}\theta$$
$$-20\pi k^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}5\sin 4\theta+4\sin 2\theta+\sin 8\theta\;\mathrm{d}\theta$$
$$=20\pi k^2\left[-\frac{5}{4}\cos 4\theta-2\cos 2\theta-\frac{1}{8}\cos 8\theta\right]_0^{\frac{\pi}{3}}$$
$$-20\pi k^2\left[-\frac{5}{4}\cos 4\theta-2\cos 2\theta-\frac{1}{8}\cos 8\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$=20\pi k^2\left[-\frac{5}{4}\left(-\frac{1}{2}\right)-2\left(-\frac{1}{2}\right)-\frac{1}{8}\left(-\frac{1}{2}\right)+\frac{5}{4}+2+\frac{1}{8}\right]$$
$$-20\pi k^2\left[-\frac{5}{4}(1)-2(-1)-\frac{1}{8}(1)+\frac{5}{4}\left(-\frac{1}{2}\right)+2\left(-\frac{1}{2}\right)+\frac{1}{8}\left(-\frac{1}{2}\right)\right]$$

	$= 20\pi k^2 \left[\dfrac{49}{8}\right]$ $= \dfrac{245}{2}\pi k^2$ $\therefore d = 245$	

10 (a) [3]	Let the eccentricity of the elliptical orbit be *e*. Let *c* be the distance between the centre of the orbit and the focus of the ellipse, and *a* be the semi-major axis.  Then, $r = a - c = a(1-e)$ and $R = a + c = a(1+e)$ Thus, $\dfrac{R}{r} = \dfrac{1+e}{1-e}$ $(1-e)R = (1+e)r$ $e(R+r) = R - r$ $e = \dfrac{R-r}{R+r}$	
(b) (i) [3]	$e = \dfrac{54000+6400-(4200+6400)}{54000+6400+(4200+6400)} \approx 0.70141 \ (\dfrac{249}{355})$ $a = \dfrac{54000+6400+4200+6400}{2} = 35500$ The height of the satellite above the earth surface when crosses the equatorial plane is $a(1-e^2) - 6400 = 35500(1-0.70141^2) - 6400 = 11635 \simeq 11600\,\text{km}$	
(b) (ii) [3]	Taking the South Pole-North Polar as the *y*-axis and the centre of the earth as its focus.  	

	Polar equation of the orbit is $r=\dfrac{a\left(1-e^{2}\right)}{1-e\sin\theta}=\dfrac{35500\left(1-0.70141^{2}\right)}{1-0.70141\sin\theta}\approx\dfrac{18035}{1-0.70141\sin\theta}$ Area swept out from perigee to equatorial plane $\dfrac{1}{2}\int_{-\frac{\pi}{2}}^{0}\left(\dfrac{18035}{1-0.70141\sin\theta}\right)^{2}\mathrm{d}\theta=131814951$ Area swept out from equatorial plane to apogee $\dfrac{1}{2}\int_{0}^{\frac{\pi}{2}}\left(\dfrac{18035}{1-0.70141\sin\theta}\right)^{2}\mathrm{d}\theta=1279181140$ Thus, the time taken for the satellite to travel from the equatorial plane to apogee is $\dfrac{1279181140}{131814951}\times0.5=4.85\,\text{hrs}$	
(c) **[4]**	Let the line $\theta=\alpha$ be the axis of the parabola, where $0<\alpha<\dfrac{\pi}{2}$. Using the line $\theta=\alpha$ as the “new” initial line of the polar coordinate system, let $\phi=\theta-\alpha$.  Then the equation of the parabola is $r=\dfrac{a}{1-\cos\phi}=\dfrac{a}{1-\cos\left(\theta-\alpha\right)}$. From the given observations: $50=\dfrac{a}{1-\cos\left(\dfrac{\pi}{3}-\alpha\right)}$ $80=\dfrac{a}{1-\cos\left(\dfrac{\pi}{4}-\alpha\right)}$ Eliminating *a*, we get $\dfrac{50}{80}=\dfrac{1-\cos\left(\dfrac{\pi}{4}-\alpha\right)}{1-\cos\left(\dfrac{\pi}{3}-\alpha\right)}$ Using GC, we obtain $\alpha=0.90097, 6.1779$. Since $0<\alpha<\dfrac{\pi}{2}$, then $\alpha=0.90097$ $a=50\left(1-\cos\left(\dfrac{\pi}{3}-0.90097\right)\right)=0.5336$ Thus, the shortest distance between the comet and the sun is $\dfrac{1}{2}a=0.267\,\text{AU}$.	