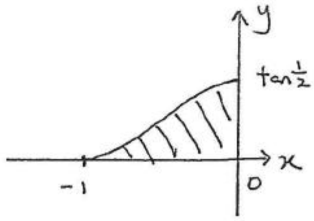
Victoria Junior College
H2 Mathematics (9740) – 2012 Preliminary Examinations P1 Solutions

1.

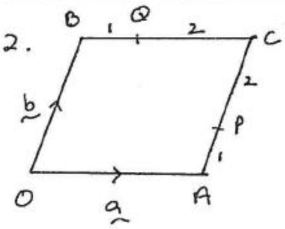


$$\text{Area} = \int_{-1}^{0} y \, dx$$

$$= \int_0^1 \left(\tan\frac{\theta}{2}\right)(2\theta)\, d\theta$$

$$= 0.3513$$

2.



(i) $\vec{QP} = \vec{QC} + \vec{CP}$

$= \frac{2}{3}\underset{\sim}{a} - \frac{2}{3}\underset{\sim}{b}$.

Alternative

$\vec{QP} = \vec{OP} - \vec{OQ}$ and apply Ratio Theorem.

(ii) $\vec{OC}\cdot\vec{QP} = (\underset{\sim}{a}+\underset{\sim}{b})\cdot\frac{2}{3}(\underset{\sim}{a}-\underset{\sim}{b})$

$= \frac{2}{3}(\underset{\sim}{a}\cdot\underset{\sim}{a} - \underset{\sim}{b}\cdot\underset{\sim}{b}) = \frac{2}{3}|\underset{\sim}{a}|^2 - \frac{2}{3}|\underset{\sim}{b}|^2$

(iii) $\vec{OC}\cdot\vec{QP} = 0 \Rightarrow |\underset{\sim}{a}| = |\underset{\sim}{b}|$

$\therefore$ OACB is a rhombus.

3(i) $\frac{1}{a+bx} = (a+bx)^{-1} = a^{-1}\left(1+\frac{b}{a}x\right)^{-1}$

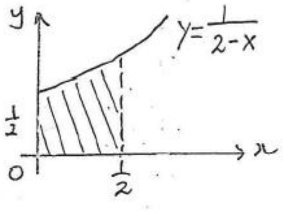
$= \frac{1}{a}\left(1 - \frac{b}{a}x + \frac{(-1)(-2)}{2!}\left(\frac{b}{a}x\right)^2\right) + \cdots$

$= \frac{1}{a} - \frac{b}{a^2}x + \frac{b^2}{a^3}x^2 + \cdots$

$\therefore \frac{1}{a} - \frac{b}{a^2}x + \frac{b^2}{a^3}x^2 = \frac{1}{2} + \frac{1}{4}x + cx^2$

$\Rightarrow a = 2, \quad b = -1, \quad c = \frac{1}{8}$.

(ii) $y = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \cdots$



$$\text{Area} \approx \int_0^{\frac{1}{2}} \frac{1}{2-x}\, dx$$

$$\approx \int_0^{\frac{1}{2}} \left(\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2\right) dx$$

$$= \left[\frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{24}\right]_0^{\frac{1}{2}} = \frac{55}{192}$$

$$\int_0^{\frac{1}{2}} \frac{1}{2-x}\, dx = \left[-\ln|2-x|\right]_0^{1/2} = -\ln\frac{3}{2} + \ln 2$$

$$= \ln 4/3.$$

$$\therefore \ln 4/3 \approx \frac{55}{192}$$

4i) Let P_n be the proposition that

$$\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}-\cdots-\frac{1}{x^n}=\frac{1}{x^n(x-1)},\ n\in\mathbb{Z}^+,\ x\notin\{0,1\}$$

Consider P_1 :

$$LHS=\frac{1}{x-1}-\frac{1}{x}=\frac{x-(x-1)}{x(x-1)}=\frac{1}{x(x-1)}$$

$$RHS=\frac{1}{x(x-1)}=LHS.$$

$\therefore P_1$ is true.

Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e

$$\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}-\cdots-\frac{1}{x^k}=\frac{1}{x^k(x-1)}$$

Consider P_{k+1},

$$LHS=\frac{1}{x-1}-\frac{1}{x}-\cdots-\frac{1}{x^k}-\frac{1}{x^{k+1}}$$

$$=\frac{1}{x^k(x-1)}-\frac{1}{x^{k+1}}$$

$$=\frac{x-(x-1)}{x^{k+1}(x-1)}=\frac{1}{x^{k+1}(x-1)}=RHS$$

$\therefore P_{k+1}$ is true.

Since P_1 is true and P_k true $\Rightarrow P_{k+1}$ true, P_n is true for all $n\in\mathbb{Z}^+$.

ii) $$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\frac{1}{x^n(x-1)}=\sum_{n=1}^{\infty}\frac{1}{(x-1)}\left(\frac{1}{x}\right)^n$$

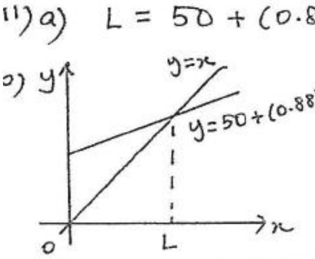
For S_∞ to exist, $\left|\frac{1}{x}\right|<1\Rightarrow|x|>1$

$\therefore x>1$ or $x<-1$

5(i) $U_n=50(0.88)^n\Rightarrow U_6=23.2$

$\therefore$ Amt of antibiotic remained $=23.2$ units

ii) a) $L=50+(0.88)^6L\Rightarrow L=93.4$

b) 

From graph,

if $x<L$, $50+(0.88)^6x>x$

if $x>L$, $50+(0.88)^6x<x$

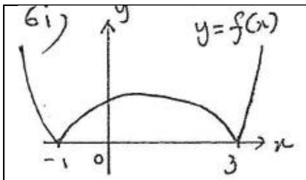
$\therefore$ If $x_n<L$, $x_{n+1}>x_n$

If $x_n>L$, $x_{n+1}<x_n$.

<u>Alternative</u> Consider $x_{n+1}-x_n=50+0.88^6x_n-x_n$

If $x_n<L$, $x_n<\frac{50}{1-0.88^6}\Rightarrow 50+0.88^6n-x_n>0$ i.e. $x_{n+1}>x_n$ //

If $x_n>L$, $x_n>\frac{50}{1-0.88^6}\Rightarrow 50+0.88^6n-x_n<0$ i.e. $x_{n+1}<x_n$ //

6i) 

$R_f = [0, \infty)$, $D_g = [-1, \infty)$

since $R_f \subseteq D_g$, gf exists.

$R_{gf} = [-2, \infty)$

ii) $f(-1) = f(3)$, $\therefore f$ is not 1-1.

largest $k = -1$.

iii) $f(x) = (x+1)(x-3)$ for $x \leq -1$.

Let $y = x^2 - 2x - 3$

$= (x-1)^2 - 4$

$x = 1 \pm \sqrt{y+4}$

$= 1 - \sqrt{y+4}$ since $x \leq -1$.

$\therefore f^{-1} : x \mapsto 1 - \sqrt{x+4}$, $x \geqslant 0$.

7ai) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots$

$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 + \cdots$

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 \ldots$

$\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = 1 + \frac{1}{3.4} + \frac{1}{5.4^2} + \cdots$

$= 2(\frac{1}{2}) + \frac{2}{3.2^3} + \frac{2}{5.2^5} + \cdots$

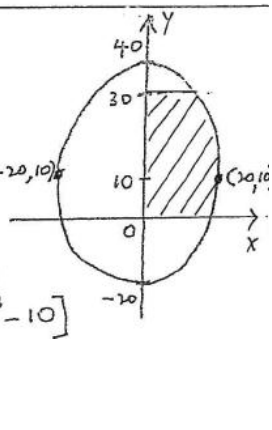
Let $x = \frac{1}{2}$, then $\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = \ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \ln 3$.

b) $\sum_{r=37}^{a} (2r)^2 = 4\left[\sum_{r=1}^{a} r^2 - \sum_{r=1}^{36} r^2\right]$

$= 4\left[\frac{a}{6}(a+1)(2a+1) - \frac{36}{6}(36+1)(72+1)\right]$

$= \frac{2}{3}a(a+1)(2a+1) - 64824$

8i) $\frac{x^2}{20^2} + \frac{(y-10)^2}{30^2} = 1$



ii) $\text{Vol} = \int_0^h \pi x^2 \, dy$

$= \pi \int_0^h 20^2 \left[1 - \frac{(y-10)^2}{30^2}\right] dy$

$= 400\pi \left[y - \frac{(y-10)^3}{900 \times 3}\right]_0^h$

$= \frac{4}{27}\pi \left[2700h - (h-10)^3 - 10\right]$

8iii) $\frac{dV}{dt} = 1000$, $\frac{dV}{dh} = \frac{4}{27}\pi\left[2700 - 3(h-10)^2\right]$

when $h = 15$, $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{1}{\left(\frac{dV}{dh}\right)}$

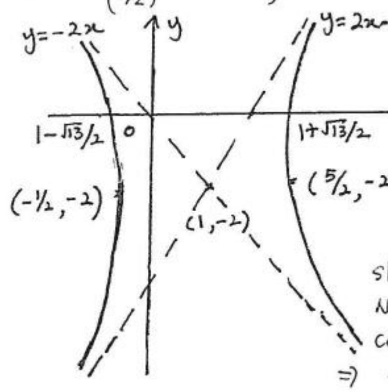
$$= 1000 \times \frac{27}{4\pi(2700 - 3\times 5^2)}$$

$$= 0.819$$

$\therefore$ the water level is increasing at 0.819 cm min^{-1} when $h = 15$.

$\frac{dh}{dt}$ is min when $h - 10 = 0$ i.e $h = 10$

9a) $\frac{(x-1)^2}{(3/2)^2} - \frac{(y+2)^2}{3^2} = 1$



$\frac{4(x-1)^2}{9} = \frac{(y+2)^2}{9}$

$y = -2 \pm 2(x-1)$

Asymptotes are $y = -2x$ or $y = 2x - 4$.

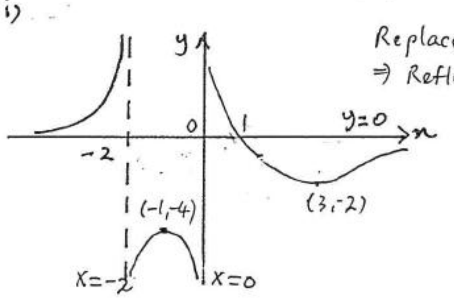
sketch the line $x = k^2$.

No real roots $\Rightarrow$ Line and curve do not intersect

$\Rightarrow -\frac{1}{2} < k^2 < \frac{5}{2}$

$\Rightarrow -\sqrt{\frac{5}{2}} < k < \sqrt{\frac{5}{2}}$

b) i)

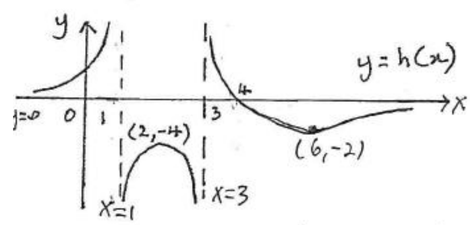


Replace x with $-x$

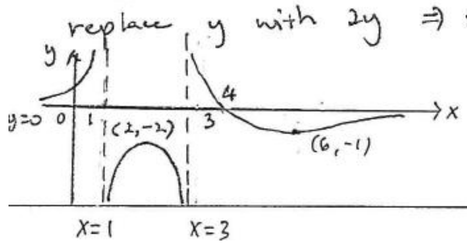
$\Rightarrow$ Reflect in the y-axis.

$y = h(x+3)$

ii) replace x with $x-3$ for i) $\Rightarrow$ translate 3 units in positive x-direction



replace y with $2y$ $\Rightarrow$ scale // y-axis with scale factor $\frac{1}{2}$



10ai) $\int \frac{1}{y} \ln y \, dy = (\ln y)^2 - \int (\ln y) \frac{1}{y} \, dy$

$\therefore 2 \int \frac{1}{y} \ln y \, dy = (\ln y)^2 + B$

$\int \frac{1}{y} \ln y \, dy = \frac{1}{2} (\ln y)^2 + c.$

ii) $\frac{dy}{dx} = \frac{y}{\ln y}$

$\int \frac{1}{y} \ln y \, dy = \int 1 \, dx$

$x = \frac{1}{2} (\ln y)^2 + c$

$\ln y = \pm \sqrt{2(x-c)}$

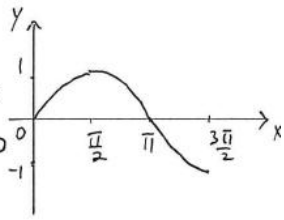
Since $y > 1$, $\ln y > 0$. $\therefore \ln y = \sqrt{2(x-c)}$

$y = e^{\sqrt{2(x-c)}}$

bi) $\frac{d}{dx} \left[x \sin(\ln x) - x \cos(\ln x) \right]$

$= x \cos(\ln x) \frac{1}{x} + \sin(\ln x) + x \sin(\ln x) \frac{1}{x} - \cos(\ln x)$

$= 2 \sin(\ln x)$



ii) $\frac{\pi}{2} \le x \le \pi$, $\sin x \ge 0$

$\pi \le x \le \frac{3\pi}{2}$, $\sin x \le 0$

$\int_{e^{\pi/2}}^{e^{3\pi/2}} |\sin(\ln x)| \, dx$

$= \int_{e^{\pi/2}}^{e^{\pi}} \sin(\ln x) \, dx - \int_{e^{\pi}}^{e^{3\pi/2}} \sin(\ln x) \, dx$

$= \frac{1}{2} \left[x \sin(\ln x) - x \cos(\ln x) \right]_{e^{\pi/2}}^{e^{\pi}} - \frac{1}{2} \left[x \sin(\ln x) - x \cos(\ln x) \right]_{e^{\pi}}^{e^{3\pi/2}}$

$= \frac{1}{2} \left[e^{\pi} - e^{\pi/2} + e^{3\pi/2} + e^{\pi} \right] = e^{\pi} - \frac{1}{2} e^{\pi/2} + \frac{1}{2} e^{3\pi/2}$

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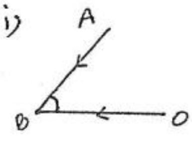
11) $\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 7$, $\underset{\sim}{r} = \begin{pmatrix} 8 \\ -2 \\ 8 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$

i)



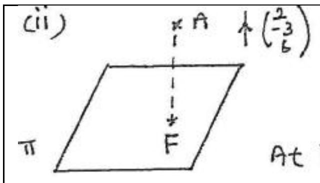
$\vec{OA} = \begin{pmatrix} 8 \\ -2 \\ 8 \end{pmatrix} + 4 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 20 \\ -6 \\ 16 \end{pmatrix}$

$\vec{OB} = \begin{pmatrix} 8 \\ -2 \\ 8 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ -4 \\ 12 \end{pmatrix}$

$\vec{AB} = \begin{pmatrix} 14 \\ -4 \\ 12 \end{pmatrix} - \begin{pmatrix} 20 \\ -6 \\ 16 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$

$\measuredangle ABO = \cos^{-1} \left[\frac{\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}}{\sqrt{56}} \cdot \frac{\begin{pmatrix} 14 \\ -4 \\ 12 \end{pmatrix}}{\sqrt{356}} \right] = 172.5°$

(ii)



Equation of l_{AF} is

$$\underset{\sim}{r} = \begin{pmatrix}20\\-6\\16\end{pmatrix} + s\begin{pmatrix}2\\-3\\6\end{pmatrix},\ s \in \mathbb{R}$$

At F, $\left[\begin{pmatrix}20\\-6\\16\end{pmatrix} + s\begin{pmatrix}2\\-3\\6\end{pmatrix}\right] \cdot \begin{pmatrix}2\\-3\\6\end{pmatrix} = 7$

$\Rightarrow 154 + 49s = 7$

$\Rightarrow s = -3$

$\therefore \overrightarrow{OF} = \begin{pmatrix}20\\-6\\16\end{pmatrix} - 3\begin{pmatrix}2\\-3\\6\end{pmatrix} = \begin{pmatrix}14\\3\\-2\end{pmatrix}$

coordinates of F is $(14, 3, -2)$.

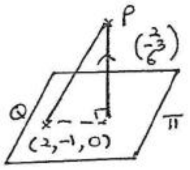
(iii) $x = 0 \Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = 0$

Let θ be the angle b/w π & $x = 0$

$$\cos\theta = \frac{\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\6\end{pmatrix}}{7} = \frac{2}{7}$$

$\theta = 73.4^\circ$ $\therefore$ The acute $\measuredangle$ is 73.4°

(iv) $\overrightarrow{OP} = \begin{pmatrix}8\\-2\\8\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$



$Q(2, -1, 0)$ is a pt in π.

$\overrightarrow{PQ} = \begin{pmatrix}2\\-1\\0\end{pmatrix} - \begin{pmatrix}8+3\lambda\\-2-\lambda\\8+2\lambda\end{pmatrix} = \begin{pmatrix}-6-3\lambda\\1+\lambda\\-8-2\lambda\end{pmatrix}$

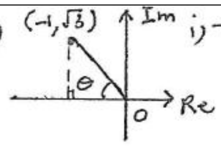
$\perp$ dist $= \left|\overrightarrow{PQ} \cdot \hat{\underset{\sim}{n}}\right| = \left|\begin{pmatrix}-6-3\lambda\\1+\lambda\\-8-2\lambda\end{pmatrix} \cdot \frac{\begin{pmatrix}2\\-3\\6\end{pmatrix}}{7}\right|$

$= \frac{1}{7}\left|63 + 21\lambda\right| = \left|9 + 3\lambda\right|$

$|9 + 3\lambda| < 3 \Rightarrow -3 < 3\lambda + 9 < 3$

$\Rightarrow -4 < \lambda < -2$

$\therefore \{\lambda \in \mathbb{R} : -4 < \lambda < -2\}$

12a) 

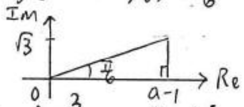
i) $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$

$\therefore \arg(z) = 2\pi/3$.

ii) $\arg(z(z+a)) = \arg z + \arg(z+a) = \frac{5\pi}{6}$

$\Rightarrow \arg(z+a) = \frac{5\pi}{6} - \frac{2\pi}{3} = \frac{\pi}{6}$

$\Rightarrow \arg(a-1+\sqrt{3}i) = \frac{\pi}{6}$



$\therefore \tan\frac{\pi}{6} = \frac{\sqrt{3}}{a-1} \Rightarrow \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{a-1}$

$\Rightarrow a = 4$

b) $iz^3 = 3-3i \Rightarrow z^3 = \frac{3-3i}{i} = -3-3i$

$z^3 = \sqrt{18}\, e^{i(-\frac{3\pi}{4}+2k\pi)}, \quad k = 0, \pm 1$

$z = 18^{1/6} e^{i(-\frac{\pi}{4}+\frac{2k}{3}\pi)} = 18^{\frac{1}{6}} e^{i(\frac{8k-3}{12})\pi}$

$= 18^{1/6} e^{i(-\pi/4)}, \ 18^{1/6} e^{i(\frac{5\pi}{12})}, \ 18^{1/6} e^{i(-\frac{11\pi}{12})}$

$-w^3 = 3-3i$

$\Rightarrow i(-iw)^3 = 3-3i$

$-iw = z \Rightarrow w = \frac{z}{-i} = iz = e^{i\frac{\pi}{2}}.z$

$\therefore w = 18^{1/6} e^{i(\pi/4)}, \ 18^{1/6} e^{i(\frac{11\pi}{12})}, \ 18^{1/6} e^{i(\frac{-5\pi}{12})}$

<u>Alternative</u>

$-w^3 = 3-3i$

$w^3 = -3+3i$

$= (-3-3i)^*$

$= (z^3)^*$

$= (z^*)^3$

$\therefore w = z^*$

$= 18^{\frac{1}{6}} e^{i\frac{\pi}{4}}, \ 18^{\frac{1}{6}} e^{-i\frac{5\pi}{12}}, \ 18^{\frac{1}{6}} e^{i\frac{11\pi}{12}}$.