Tutorial 8C: Vectors III (Lines in Three-Dimensional Space)

Basic Mastery Questions

1. Find a vector equation for each of the following lines:
   (a) a line which passes through the point with coordinates (1, 2, −3) and is parallel to the vector $3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$;
   (b) a line which passes through the points (2, 0, 6) and (−3, 5, 1).

(a) $l: \ \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$, where $\lambda \in \mathbb{R}$

(b) $\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\quad \therefore$ Ans: $l: \ \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, where $\mu \in \mathbb{R}$

2. Convert the following vector equations to Cartesian equations.
   (i) $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$ (ii) $\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}, \ \mu \in \mathbb{R}$.

(i) $x - 2 = 3 - y = \dfrac{z+1}{2}$

(ii) $\dfrac{x}{3} = \dfrac{z}{5}, \ y = 4$
$\Rightarrow 5x = 3z, \ y = 4$

3. Convert the following Cartesian equations of straight lines to vector equations.
   (i) $\dfrac{1-x}{5} = \dfrac{y}{5} = z - 1$ (ii) $\dfrac{2x+1}{2} = y, \ z = 5$

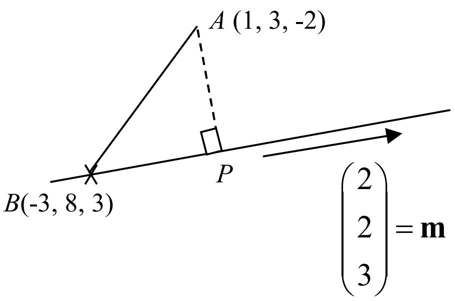
(i) $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}$, where $\lambda \in \mathbb{R}$

(ii) $\mathbf{r} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, where $\lambda \in \mathbb{R}$

4. **(N07/1/7)** The point P is the foot of the perpendicular from the point $A\ (1,3,-2)$ to the line given by $\frac{x+3}{2}=\frac{y-8}{2}=\frac{z-3}{3}$.
Find the coordinates of P, and hence find the length of AP.

[solution]

Vector equation of this line is $\mathbf{r}=\begin{pmatrix}-3\\8\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\3\end{pmatrix}$, where $\lambda\in\mathbb{R}$



Let B be the point $(-3,8,3)$ which lies on the line.

$$\overrightarrow{BA}=\begin{pmatrix}1\\3\\-2\end{pmatrix}-\begin{pmatrix}-3\\8\\3\end{pmatrix}=\begin{pmatrix}4\\-5\\-5\end{pmatrix}$$

$$\overrightarrow{BP}=\left(\overrightarrow{BA}\cdot\widehat{\mathbf{m}}\right)\widehat{\mathbf{m}}$$

$$\Rightarrow\overrightarrow{OP}=\overrightarrow{OB}+\left(\overrightarrow{BA}\cdot\widehat{\mathbf{m}}\right)\widehat{\mathbf{m}}$$

$$=\begin{pmatrix}-3\\8\\3\end{pmatrix}+\left[\begin{pmatrix}4\\-5\\-5\end{pmatrix}\cdot\frac{1}{\sqrt{17}}\begin{pmatrix}2\\2\\3\end{pmatrix}\right]\frac{1}{\sqrt{17}}\begin{pmatrix}2\\2\\3\end{pmatrix}$$

$$=\begin{pmatrix}-3\\8\\3\end{pmatrix}+\frac{8-10-15}{17}\begin{pmatrix}2\\2\\3\end{pmatrix}$$

$$=\begin{pmatrix}-5\\6\\0\end{pmatrix}$$

$\therefore P(-5,6,0)$

Hence, $AP=\left|\overrightarrow{OP}-\overrightarrow{OA}\right|=\left|\begin{pmatrix}-5\\6\\0\end{pmatrix}-\begin{pmatrix}1\\3\\-2\end{pmatrix}\right|=\left|\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|=\sqrt{36+9+4}=7$ units