RAFFLES INSTITUTION
2023 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 23

MATHEMATICS **9758/02**

Paper 2 **3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE							
SECTION A: PURE MATHEMATICS							TOTAL
Q1	Q2	Q3	Q4			Total	
9	9	11	11			40	
SECTION B: PROBABILITY AND STATISTICS							
Q5	Q6	Q7	Q8	Q9	Q10	Total	
6	8	9	12	12	13	60	100

This document consists of ~~**22**~~ 6 printed pages~~ and **2** blank pages~~.

RAFFLES INSTITUTION
Mathematics Department

Section A: Pure Mathematics [40 marks]

1 The curve C is defined by the parametric equations

$$x = t^2 - t, \quad y = 3t + 2 \qquad \text{where } t \in \mathbb{R}.$$

(a) Show algebraically that $x \geq -\frac{1}{4}$ for all values of t. [2]

The line N is the normal to C at the point where $t = -1$.

(b) Find the cartesian equation of N. [3]

(c) Find the area of the finite region enclosed by C, N and $x = -\frac{1}{4}$, correct to 3 decimal places. [4]

2 The terms of an arithmetic progression 1, 4, 7, 10, …, are grouped into sets containing $1, 3, 5, 7, \ldots$ integers, as indicated below, so that the number of integers in each set after the first is two more than the number of integers in the previous set.

$$\{1\},\ \{4, 7, 10\},\ \{13, 16, 19, 22, 25\},\ \{28, 31, 34, 37, 40, 43, 46\},\ \ldots.$$

(a) Find the total number of integers in the first n sets, in terms of n. [2]

(b) Find the last integer in the nth set, in terms of n. Hence, find the value of k, given that the integer 2023 occurs in the kth set. [4]

(c) Find the sum of all the integers in the 5th to 10th set. [3]

3 It is given that $y = \mathrm{e}^x \cos 3x$.

(a) Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\frac{\mathrm{d}y}{\mathrm{d}x} - 10y$. Hence find the first four non-zero terms of the Maclaurin expansion of $\mathrm{e}^x \cos 3x$. [6]

(b) Verify that the same result in part **(a)** is obtained if the standard series expansions for e^x and $\cos x$ are used. [2]

(c) Show that $\ln(1 + \mathrm{e}^x \cos 3x)$ can be expressed as $\ln 2 + \frac{1}{2}x - \frac{17}{8}x^2 + \ldots$. [3]

4 With reference to the origin O, the points A and B are such that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$, where $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-parallel vectors.

(a) Show that the vector

$$\mathbf{n} = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\,\mathbf{b}$$

is perpendicular to $\mathbf{a}$ and is parallel to the plane OAB. [4]

The vectors $\mathbf{a}$ and $\mathbf{b}$ are now given by

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

(b) Find a unit vector $\mathbf{m}$ which is perpendicular to $\mathbf{a}$ and is parallel to plane OAB. [2]

(c) Find the coordinates of the point of intersection between the line OA and the line which passes through point B and is parallel to $\mathbf{m}$. [2]

(d) Hence, or otherwise, find an equation of the line which is a reflection of the line OB in the line OA. [3]

Section B: Probability and Statistics [60 marks]

5 In this question you should state the parameters of any normal distributions you use.

Griffles has a large collection of small massage balls. The mass of the small massage balls, in grams, are normally distributed with a mean mass of 200 grams. The mass of 98.273% of these small massage balls is between 195 grams and 205 grams.

(a) Show that the standard deviation of the mass of the small massage balls is 2.1 grams, correct to 1 decimal place. [2]

Griffles has another large collection of medium massage balls. The mass of the medium massage balls, in grams, have the distribution $\mathrm{N}\left(500, 1.4^2\right)$.

(b) Find the probability that the total mass of 6 small massage balls exceeds twice the mass of a medium massage ball by more than 210 grams. [3]

(c) State an assumption needed for your calculation in part **(b)**. [1]

6 A group of 5 boys and 3 girls sit at random at a round table. Find the number of arrangements so that

(a) no 2 girls are adjacent to each other, [3]

(b) all 3 girls are seated together, [3]

(c) exactly 2 of the 3 girls are adjacent to each other. [2]

7 For two independent events A and B, it is given that $\mathrm{P}(A) = a$ and $\mathrm{P}(B) = b$.

(a) Show that A' and B' are independent events. [2]

It is given that $\mathrm{P}(A \cup B) = 0.7$ and $b = 0.5$.

(b) Show that $a = 0.4$. [2]

For a third event C, it is given that $\mathrm{P}(C) = 0.3$ and events A and C are independent.

(c) If events B and C are mutually exclusive, find $\mathrm{P}\left(A' \cap B' \cap C'\right)$. [2]

(d) If events B and C are not mutually exclusive, find the greatest possible value of $\mathrm{P}\left(A' \cap B' \cap C'\right)$, showing your working clearly. [3]

8 From a school-wide survey done in 2022, it was found that the students in a certain school spend an average of 45.5 hours per week on social media platforms. In 2023, the school's Student Well-being team decides to carry out a survey on a group of 50 randomly chosen students.

(a) State what it means for a sample to be random in this context. [1]

The time, x hours, spent per week on social media platforms by the 50 students are summarised below.

$$\sum(x - 45.5) = 55 \qquad \sum(x - 45.5)^2 = 1494.74$$

(b) Calculate unbiased estimates of the population mean and variance of the time spent by students per week on social media platforms. [2]

(c) State hypotheses that can be used to test if the mean time spent by students per week differs from 45.5 hours. Work out the critical region in this case, and use it to carry out the test at the 5% level of significance, giving your conclusion in the context of the question. [4]

(d) Explain why it is not necessary to assume that the time spent by students per week on social media platforms follows a normal distribution when performing the test in part **(c)**. [1]

It is now given that the time spent by students per week on social media platforms follows a normal distribution with a variance of 25 hours^2. Based on observations, the Head of Student Well-being, Ms Tan, believes that the mean time spent by students per week on social media platforms has in fact increased.

(e) By carrying out the test at α% level of significance using data collected from 50 randomly chosen students with a mean of 47 hours, determine the set of values of α for which Ms Tan's belief should be accepted. [4]

9 In a study to determine the relationship between the temperature T (measured in degrees Celsius) and the altitude H (height above sea level, measured in metres) for a certain mountain, a team of meteorologists ascended and descended the mountain on a day with clear weather and obtained readings of T and H.

The table below shows the 10 readings obtained:

H	2300	2390	2626	2631	2700	3126	3131	3400	3450	3580
T	11.80	9.26	10.29	7.99	9.52	5.20	7.26	3.72	4.43	2.31

(a) On the same axes, sketch a scatter diagram of the data and draw the regression line of T against H. [2]

(b) State the product moment correlation coefficient between T and H. Comment, in context, on the value obtained. [2]

It was found that 5 of these 10 readings were recorded between 2 am and 5.30 am when the team of meteorologists were ascending the mountain, while the other 5 were recorded between 8 am and 11.30 am when they were descending. The routes used for ascending and descending were the same. Between 5.30 am and 8 am, the meteorologists remained on the summit.

For these readings, the regression equation for ascending, corrected to 3 significance figures, is $T = 23.1 - 0.00574H$. The regression equation for descending is $T = a + bH$.

(c) Find, correct to 3 significance figures, the values of a and b. [2]

One of the meteorologists recorded the temperature near the foot of the mountain as 26 degrees Celsius.

(d) Determine whether it is suitable to use any of the regression equations above to estimate the altitude of the foot of the mountain. [1]

A temperature of F degrees Fahrenheit is equivalent to a temperature of T degrees Celsius, where $F = \frac{9}{5}T + 32$.

(e) The regression equation for ascending, $T = 23.1 - 0.00574H$ can be written in the form $F = c + dH$, where c and d are constants. Find the values of c and d. [2]

(f) Use the equation found in part **(e)** to estimate the temperature in degrees Fahrenheit when the meteorologists were at an altitude of 2800 m while ascending. Give two reasons why you would expect this estimate to be reliable. [3]

10 Two fair dice are thrown and the product of the two numbers on the top faces is computed. Let R_n be the remainder when the product is divided by n.

(a) Find the probability distribution of R_2 and hence find $\mathrm{E}(R_2)$. [3]

(b) Show that $\mathrm{P}(R_6 = 0) = \dfrac{5}{12}$. [1]

The two dice are thrown m times and the product of the two numbers on the top faces are computed each time. X is the number of products that are divisible by 6 after m throws.

(c) Show that $\mathrm{P}(X = 2) = Am(m-1)\left(\dfrac{7}{12}\right)^m$, where A is a constant to be determined.

Given that $\mathrm{P}(X = 2) = 0.03604$, correct to 5 decimal places, deduce the value of m, justifying your answer. [4]

(d) Class C has 35 students and class D has 30 students. Each student in the two classes throws the two dice 10 times. Estimate the probability that the average number of products that are divisible by 6 obtained by a student in class C exceeds the average number of products that are divisible by 6 obtained by a student in class D by more than 0.2. [5]