2015 IJC H2 Maths Prelim 2 Paper 1 (Suggested Solution)

1 A water tank is in the shape of a vertically inverted cone with height 2 m and based radius 0.5 m. Water is flowing into the tank at a constant rate of 0.01 $\text{m}^3\ \text{s}^{-1}$. At the same time, water is leaking out of the tank at a constant rate of $1.2\times10^{-4}\ \text{m}^3\ \text{s}^{-1}$ through a small hole at the vertex.. Find the rate of change of the depth of the water level in the tank at the instant when the depth of the water is 0.5 m. [4]

Q1	Solutions
	Let the radius of the water level be r m and the depth of the water level be h m at time t seconds. Using similar triangles, $$\frac{r}{h}=\frac{0.5}{2}=\frac{1}{4}, \Rightarrow r=\frac{h}{4}$$ $\frac{\mathrm{d}V}{\mathrm{d}t}=0.01-0.00012=0.00988$ $V=\frac{1}{3}\pi r^2 h=\frac{1}{48}\pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{16}\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}$ When $h=0.5$, $0.00988=\frac{1}{16}\pi(0.5)^2\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{0.00988\times16}{0.25\pi}=0.201$ (3 s.f.) $\therefore$ the rate of change of the depth of the water level in the tank at the instant when the depth of the water is 0.5 m is 0.201 m s^{-1}.

2 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point M is on AB produced such that $AM : BM = 4:1$ and the point N is on OB such that $ON : NB = 2:3$.

(i) Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) By considering cross product, find the ratio of the area of triangle ANB to the area of triangle OAM. [4]

Q2	**Solution**
(i)	$\overrightarrow{OB} = \dfrac{\overrightarrow{OA} + 3\overrightarrow{OM}}{4}$ $\overrightarrow{OM} = \dfrac{4\overrightarrow{OB} - \overrightarrow{OA}}{3} = \dfrac{4\mathbf{b} - \mathbf{a}}{3}$
(ii)	Area of $\Delta OAM = \dfrac{1}{2}\left\|\overrightarrow{OA} \times \overrightarrow{OM}\right\|$ $= \dfrac{1}{2}\left\|\mathbf{a} \times \dfrac{4\mathbf{b} - \mathbf{a}}{3}\right\|$ $= \dfrac{1}{6}\left\|\mathbf{a} \times 4\mathbf{b} - \mathbf{a} \times \mathbf{a}\right\| \quad \because (\mathbf{a} \times \mathbf{a}) = \mathbf{0}$ $= \dfrac{2}{3}\left\|\mathbf{a} \times \mathbf{b}\right\|$ $\overrightarrow{ON} = \dfrac{2}{5}\mathbf{b}$ $= \dfrac{1}{2}\left\|\overrightarrow{AN} \times \overrightarrow{AB}\right\|$ $= \dfrac{1}{2}\left\|\left(\dfrac{2}{5}\mathbf{b} - \mathbf{a}\right) \times (\mathbf{b} - \mathbf{a})\right\|$ Area of $\Delta ANB = \dfrac{1}{2}\left\|-\dfrac{2}{5}(\mathbf{b} \times \mathbf{a}) - (\mathbf{a} \times \mathbf{b})\right\| \quad \because (\mathbf{a} \times \mathbf{a})$ and $(\mathbf{b} \times \mathbf{b}) = \mathbf{0}$ $= \dfrac{1}{2}\left\|\dfrac{3}{5}(\mathbf{a} \times \mathbf{b})\right\|$ $= \dfrac{3}{10}\left\|\mathbf{a} \times \mathbf{b}\right\|$ Ratio of ΔANB to ΔOAM = 9 : 20

3 **(i)** Express $\dfrac{1}{r-1}+\dfrac{2}{r}-\dfrac{3}{r+1}$ in the form $\dfrac{Ar+B}{r(r-1)(r+1)}$, where A and B are constants to be determined . [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{n}\frac{2r-1}{r(r-1)(r+1)}$. (There is no need to express your answer as a single fraction.) [3]

(iii) Deduce that $\displaystyle\sum_{r=2}^{n}\frac{2r-1}{r(r-1)(r+1)}<\frac{5}{4}$. [1]

(iv) Using your answer to part **(ii)**, find $\displaystyle\sum_{r=4}^{n+1}\frac{2r-3}{r(r-1)(r-2)}$. [2]

Q3	Solution
(i)	$\dfrac{1}{r-1}+\dfrac{2}{r}-\dfrac{3}{r+1}$ $=\dfrac{(r)(r+1)+2(r-1)(r+1)-3r(r-1)}{r(r-1)(r+1)}$ $=\dfrac{r^2+r+2r^2-2-3r^2+3r}{r(r-1)(r+1)}$ $=\dfrac{4r-2}{r(r-1)(r+1)}$ $A=4,\ B=-2$
(ii)	$\displaystyle\sum_{r=2}^{n}\frac{2r-1}{(r-1)r(r+1)}=\frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1}+\frac{2}{r}-\frac{3}{r+1}\right)$ $=\dfrac{1}{2}\Big[\dfrac{1}{1}+\dfrac{2}{2}-\cancel{\dfrac{3}{3}}$ $+\dfrac{1}{2}+\cancel{\dfrac{2}{3}}-\cancel{\dfrac{3}{4}}$ $+\cancel{\dfrac{1}{3}}+\cancel{\dfrac{2}{4}}-\cancel{\dfrac{3}{5}}$ $+\ldots$ $+\ldots$ $+\cancel{\dfrac{1}{n-3}}+\cancel{\dfrac{2}{n-2}}-\cancel{\dfrac{3}{n-1}}$ $+\cancel{\dfrac{1}{n-2}}+\cancel{\dfrac{2}{n-1}}-\dfrac{3}{n}$ $+\cancel{\dfrac{1}{n-1}}+\dfrac{2}{n}-\dfrac{3}{n+1}\Big]$ $=\dfrac{1}{2}\left[2+\dfrac{1}{2}-\dfrac{3}{n}+\dfrac{2}{n}-\dfrac{3}{n+1}\right]$ $=\dfrac{1}{2}\left[\dfrac{5}{2}-\dfrac{1}{n}-\dfrac{3}{n+1}\right]$ $=\dfrac{5}{4}-\dfrac{1}{2n}-\dfrac{3}{2(n+1)}$

(iii)	Since $n>0$, $\frac{1}{2n}>0$ and $\frac{3}{2(n+1)}>0$. Hence, $\sum_{r=2}^{n}\frac{2r-1}{r(r-1)(r+1)}$ $=\frac{5}{4}-\frac{1}{2n}-\frac{3}{2(n+1)}<\frac{5}{4}$
(iv)	From (ii), $\sum_{r=2}^{n}\frac{2r-1}{r\left(r-1\right)\left(r+1\right)}=\frac{5}{4}-\frac{1}{2n}-\frac{3}{2(n+1)}$ To find $\sum_{r=4}^{n+1}\frac{2r-3}{r(r-1)(r-2)}$, replace r by $r+1$. $\sum_{r=4}^{n+1}\frac{2r-3}{r(r-1)(r-2)}=\sum_{r=3}^{n}\frac{2r-1}{r\left(r-1\right)\left(r+1\right)}$ $=\left[\frac{5}{4}-\frac{1}{2n}-\frac{3}{2(n+1)}\right]-\frac{3}{(1)(2)(3)}$ $\sum_{r=4}^{n+1}\frac{2r-3}{r(r-1)(r-2)}=\frac{3}{4}-\frac{1}{2n}-\frac{3}{2(n+1)}$

4 A sequence $u_1,\ u_2,\ u_3,\ldots$ is given by

$$u_1 = 1 \text{ and } u_{n+1} = \frac{4u_n - 1}{8} \text{ for } n \geq 1.$$

(i) Find the values of u_2 and u_3. [1]

(ii) It is given that $u_n \to l$ as $n \to \infty$. Showing your working, find the exact value of l. [1]

(iii) For this value of l, use the method of mathematical induction to prove that

$$u_n = 5\left(\frac{1}{2}\right)^{n+1} + l.$$ [4]

(iv) Hence find $\sum_{n=1}^{N} u_n$. [3]

Q4	Solution
(i)	$u_2 = \frac{4u_1 - 1}{8} = \frac{4(1) - 1}{8} = \frac{3}{8}$ (or 0.375) $u_3 = \frac{4u_2 - 1}{8} = \frac{4\left(\frac{3}{8}\right) - 1}{8} = \frac{1}{16}$ (or 0.0625)
(ii)	$u_n \to l$ as $n \to \infty \Rightarrow u_{n+1} \to l$ as $n \to \infty$ $\because u_{n+1} = \frac{4u_n - 1}{8}$, as $n \to \infty$, $l = \frac{4l - 1}{8}$ $8l - 4l = -1$ $l = -\frac{1}{4}$
(iii)	Let P_n be the statement $u_n = 5\left(\frac{1}{2}\right)^{n+1} - \frac{1}{4}$ for $n \in \mathbb{Z}^+$. LHS of $P_1 = u_1 = 1$ (given) RHS of $P_1 = 5\left(\frac{1}{2}\right)^2 - \frac{1}{4} = \frac{5}{4} - \frac{1}{4} = 1$ $\therefore P_1$ is true. Assume that P_k is true for some $k \in \mathbb{Z}^+$, ie. $u_k = 5\left(\frac{1}{2}\right)^{k+1} - \frac{1}{4}$. Need to prove P_{k+1}, ie $u_{k+1} = 5\left(\frac{1}{2}\right)^{k+2} - \frac{1}{4}$

	LHS = u_{k+1} $= \frac{4u_k - 1}{8}$ $= \frac{1}{2}u_k - \frac{1}{8}$ $= \frac{1}{2}\left[5\left(\frac{1}{2}\right)^{k+1} - \frac{1}{4}\right] - \frac{1}{8}$ $= 5\left(\frac{1}{2}\right)^{k+2} - \frac{1}{8} - \frac{1}{8}$ $= 5\left(\frac{1}{2}\right)^{k+2} - \frac{1}{4}$ $=$ RHS $\therefore P_{k+1}$ is true Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, $\therefore$ by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.
(iv)	$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N}\left(5\left(\frac{1}{2}\right)^{n+1} - \frac{1}{4}\right)$ $= \frac{5\left(\frac{1}{4}\right)\left[1-\left(\frac{1}{2}\right)^N\right]}{1-\frac{1}{2}} - \frac{1}{4}N$ $= \frac{5}{2}\left(1 - \frac{1}{2^N}\right) - \frac{1}{4}N$ or $\frac{5}{2} - \frac{5}{2^{N+1}} - \frac{1}{4}N$ or equivalent

5 A graphic calculator is **not** to be used in answering this question.

The complex number z is given by $2+\mathrm{i}\sqrt{3}$.

(i) Find z^4 in the form $x+\mathrm{i}y$, showing your working. [2]

(ii) Given that z is a root of the equation $2w^4+aw^2+bw+49=0$, find the values of the real numbers a and b. [3]

(iii) Using these values of a and b, find all the roots of this equation in exact form. [4]

Q5	**Solution**
(i)	$z^4=\left(2+\sqrt{3}\mathrm{i}\right)^4$ $=2^4+{}^4C_1(2)^3\left(\sqrt{3}\mathrm{i}\right)+{}^4C_2(2)^2\left(\sqrt{3}\mathrm{i}\right)^2$ $+{}^4C_3(2)\left(\sqrt{3}\mathrm{i}\right)^3+\left(\sqrt{3}\mathrm{i}\right)^4$ $=16+32\sqrt{3}\mathrm{i}-72-24\sqrt{3}\mathrm{i}+9$ $=-47+8\sqrt{3}\mathrm{i}$
(ii)	<u>Method 1</u> $2w^4+aw^2+bw+49=0$ $2\left(-47+8\sqrt{3}\mathrm{i}\right)+a\left(2+\sqrt{3}\mathrm{i}\right)^2+b\left(2+\sqrt{3}\mathrm{i}\right)+49=0$ $-94+16\sqrt{3}\mathrm{i}+4a+a\left(4\sqrt{3}\mathrm{i}\right)-3a+2b+b\sqrt{3}\mathrm{i}+49=0$ $-45+a+2b+(4a+b+16)\sqrt{3}\mathrm{i}=0$ Comparing the coefficient of real part $a+2b=45------(1)$ Comparing the coefficient of imaginary part $4a+b=-16-------(2)$ $a=-11,\ b=28$ <u>Method 2</u> Since the coefficients of the equation are real, $z^*=2-\sqrt{3}\mathrm{i}$ is also a root $\left(w-\left(2+\sqrt{3}\mathrm{i}\right)\right)\left(w-\left(2-\sqrt{3}\mathrm{i}\right)\right)$ $\left((w-2)-\sqrt{3}\mathrm{i}\right)\left((w-2)+\sqrt{3}\mathrm{i}\right)$ $=(w-2)^2-\left(\sqrt{3}\mathrm{i}\right)^2$ $=w^2-4w+7$

	$2w^4+aw^2+bw+49=\left(w^2-4w+7\right)\left(2w^2+pw+7\right)$ By comparing coeff of w^3 , (or inspection mtd) $p-8=0$ $p=8$ comparing coeff of w^2 : $a=7-32+14=-11$ comparing coeff of w: $b=56-28=28$
	<u>Method 1</u> Since the coefficients of the equation are real, $z^*=2-\sqrt{3}\text{i}$ is also a root. $\left(w-\left(2+\sqrt{3}\text{i}\right)\right)\left(w-\left(2-\sqrt{3}\text{i}\right)\right)$ $\left(\left(w-2\right)-\sqrt{3}\text{i}\right)\left(\left(w-2\right)+\sqrt{3}\text{i}\right)$ $=\left(w-2\right)^2-\left(\sqrt{3}\text{i}\right)^2$ $=w^2-4w+7$ By long division or inspection, $2w^4-11w^2+28w+49=\left(w^2-4w+7\right)\left(2w^2+8w+7\right)$ $2w^2+8w+7=0$ $w=\dfrac{-8\pm\sqrt{64-4(2)(7)}}{2(2)}$ The other roots are $2-\sqrt{3}\text{i},-2\pm\dfrac{\sqrt{2}}{2}$ **Method 2** $2w^2+8w+7=0$ $w=\dfrac{-8\pm\sqrt{64-4(2)(7)}}{2(2)}=-2\pm\dfrac{\sqrt{2}}{2}$ The other roots are $2-\sqrt{3}\text{i},-2\pm\dfrac{\sqrt{2}}{2}$

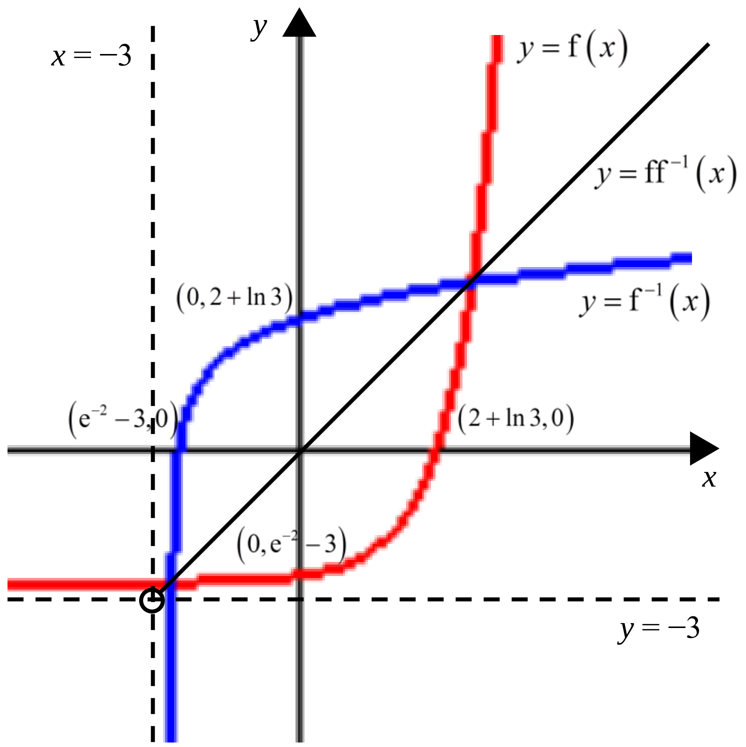
6 The function f is defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^{x-2} - 3, \quad x \in \mathbb{R} .$$

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain and range of f^{-1}. [3]

(ii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}\,\mathrm{f}^{-1}(x)$, giving the equations of any asymptotes and the exact coordinates of any points where the curves cross the x- and y-axes. [4]

(iii) Find the area of the region bounded between $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, giving your answer correct to 2 decimal places. [2]

Q6	Solutions
(i)	Let $y = \mathrm{e}^{x-2} - 3$ $y + 3 = \mathrm{e}^{x-2}$ $\ln(y+3) = x - 2$ $x = \ln(y+3) + 2$ $\mathrm{f}^{-1}(x) = \ln(x+3) + 2$ $\mathrm{f}^{-1} : x \mapsto \ln(x+3) + 2,\ x \in \mathbb{R},\ x > -3$ $\mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D}_{\mathrm{f}} = \mathbb{R}$
(ii)	
(iii)	x – coordinates of points of intersection:

	Using GC, $x = -2.9932$ or 3.9368 Area required $= \int_{-2.9932}^{3.9368} \left(\ln(x+3)+2\right)-\left(e^{x-2}-3\right)\ \mathrm{d}x$ $= 34.26$ (2 d.p.)

7 The curve C_1 has equation $y=\dfrac{1}{2a-x}+\dfrac{1}{x}$, where a is a positive constant.

(i) By using differentiation, or otherwise, find the coordinates of the stationary point and determine the nature of the stationary point. [4]

(ii) Sketch C_1, indicating clearly the equations of any asymptotes and the coordinates of the stationary point. [2]

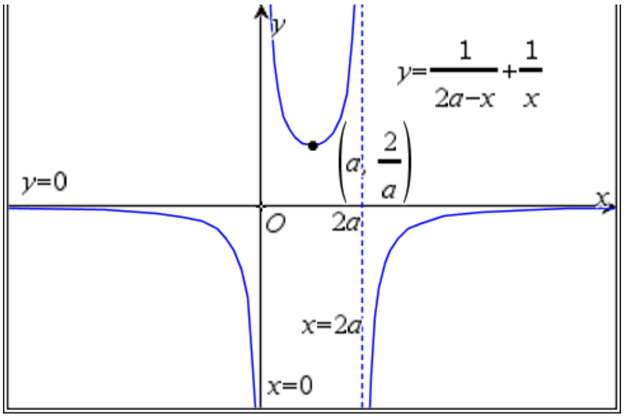
Another curve C_2 has equation $\dfrac{(x-a)^2}{a^2}+\dfrac{y^2}{b^2}=1$, where b is a positive constant.

(iii) By considering the graph of C_2, find an inequality satisfied by b such that C_1 and C_2 intersect at more than one point. [2]

(iv) Given that $a=2$ and $b=6$, find the range of values of x such that

$$\frac{1}{2a-x}+\frac{1}{x} > b\left[\sqrt{\left(1-\frac{(x-a)^2}{a^2}\right)}\right].$$ [2]

Q7	Solutions
(i)	**<u>Method 1: Using Differentiation</u>** $\dfrac{dy}{dx}=\dfrac{1}{(2a-x)^2}-\dfrac{1}{x^2}$ Putting $\dfrac{dy}{dx}=0,\ \dfrac{1}{(2a-x)^2}=\dfrac{1}{x^2}$ $(2a-x)^2=x^2 \quad \Rightarrow \quad x=\pm(2a-x)$ $x=a$ $y=\dfrac{1}{2a-a}+\dfrac{1}{a}=\dfrac{2}{a}$ $\dfrac{d^2y}{dx^2}=\dfrac{2}{(2a-x)^3}+\dfrac{2}{x^3}$ When $x=a$, $\dfrac{d^2y}{dx^2}=\dfrac{2}{a^3}+\dfrac{2}{a^3}>0$ $\left(a\ ,\ \dfrac{2}{a}\right)$ is a min point. **<u>Method 2: non-differentiation method</u>** $y=\dfrac{1}{2a-x}+\dfrac{1}{x}=\dfrac{2a}{x(2a-x)}$ is the reciprocal graph of quadratic eqn $y=x(2a-x)$ and multiplied by scale factor $2a$ in the y-direction. Since the graph of $y=x(2a-x)$ is a parabola with a max point at

<table>
<tr><td></td><td>$(a\,, a^2)$, the curve C_1 will have a min point at $x = a$, and $y = 2a\left(\frac{1}{a^2}\right) = \frac{2}{a}$.
Thus $\left(a\ ,\ \dfrac{2}{a}\right)$ is a min point.</td></tr>
<tr><td>(ii)</td><td>


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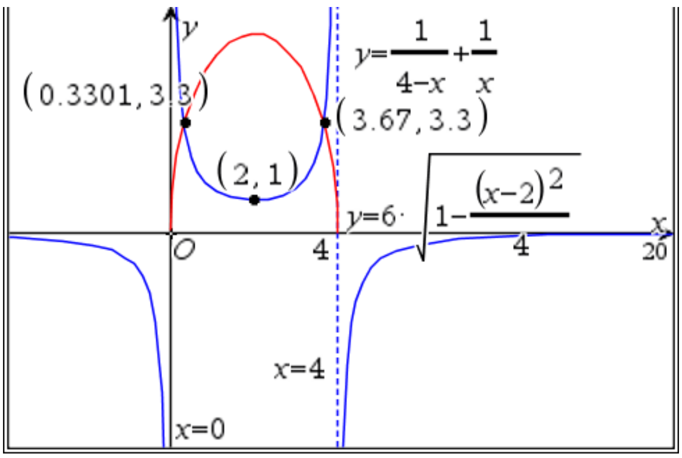
<tr><td>(iii)</td><td>$\dfrac{(x-a)^2}{a^2}+\dfrac{y^2}{b^2}=1$. The graph of C_2 is an ellipse with centre $(a, 0)$, with 2 end-pts at (0,0) and (2a,0).
For the ellipse to intersect C_1 at more than one pt, $b > \frac{2}{a}$.</td></tr>
<tr><td>7(iv)</td><td>$a=2$ and $b=6$
$\dfrac{1}{2a-x}+\dfrac{1}{x} > b\left[\sqrt{\left(1-\dfrac{(x-a)^2}{a^2}\right)}\right]$


From the graph,
$0 < x < 0.330$ or $3.67 < x < 4$ (3 s.f.)</td></tr>
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8 **(a)** A curve with equation $y = \mathrm{f}(x)$ undergoes the following sequence of transformation.

A: A translation of 3 units in the positive *y*-direction.

B: A translation of 2 units in the positive *x*-direction.

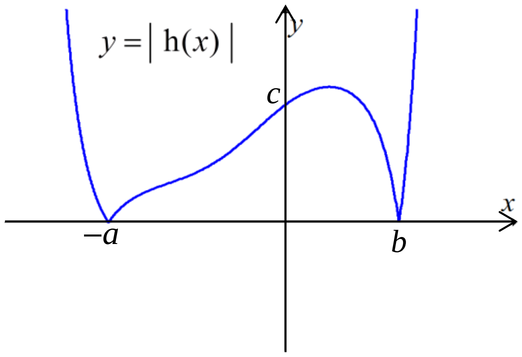
C: A stretch with scale factor 2 parallel to the *x*-axis.

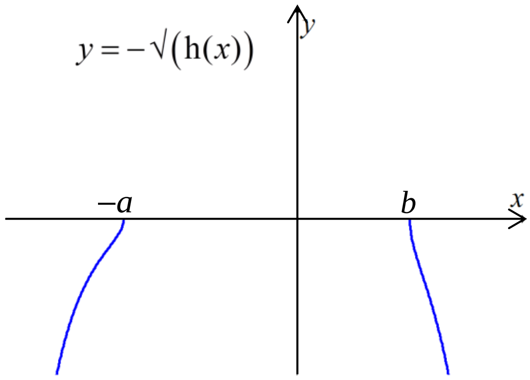
The resulting curve has equation $y = \mathrm{g}(x)$ where $\mathrm{g}(x) = 2 - \dfrac{4}{3x-7}$. Find $\mathrm{f}(x)$ in terms of *x*. [3]

Another curve with equation $y = \dfrac{1}{\mathrm{f}(x)}$ undergoes the same sequence of transformations *A*, *B* and *C*. Will the equation of the resulting curve be $y = \dfrac{1}{\mathrm{g}(x)}$? Justify your answer. [1]

Q8	**Solution**
(a)	$\mathrm{f}\left(\frac{1}{2}x-2\right)+3=2-\frac{4}{3x-7}$ $\mathrm{f}\left(\frac{1}{2}x-2\right)=-1-\frac{4}{3x-7}$ $\mathrm{f}\left(\frac{1}{2}x-2\right)=-1-\frac{4}{6\left(\frac{1}{2}x-2\right)+5}$ $\mathrm{f}(x)=-1-\frac{4}{6x+5}$ **<u>Alternatively,</u>** Since $\mathrm{g}(x)=2-\frac{4}{3x-7}$ $\mathrm{f}(x)=\mathrm{g}\left[2\left(x+2\right)\right]-3=\left[2-\frac{4}{3(2x+4)-7}\right]-3$ $=-1-\frac{4}{6x+5}$ No, it will not be $\frac{1}{g(x)}$. Undergoing the sequence of transformation, $\frac{1}{\mathrm{f}(x)}$ would become $\frac{1}{\mathrm{f}(\frac{1}{2}x-2)}+3$, but $\frac{1}{g(x)}=\frac{1}{\mathrm{f}\left(\frac{1}{2}x-2\right)+3}$. So they are different.

(b)



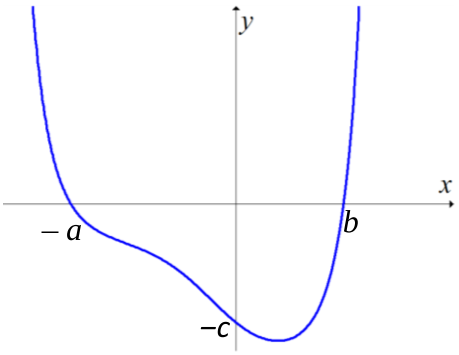
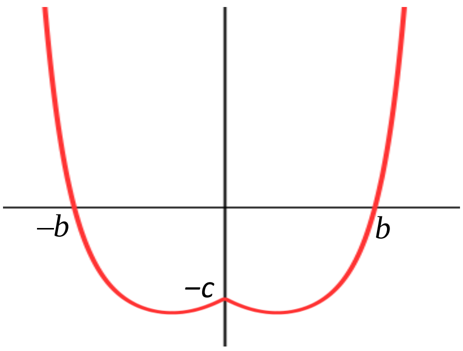


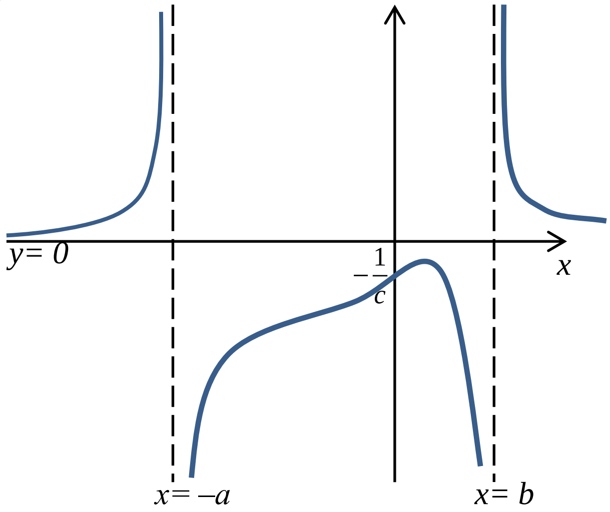
The diagrams show the graphs of $y = |\mathrm{h}(x)|$ and $y = -\sqrt{(\mathrm{h}(x))}$, where a, b and c are positive constants. Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{h}(x)$, [2]

(ii) $y = \mathrm{h}(|x|)$, [2]

(iii) $y = \dfrac{1}{\mathrm{h}(x)}$. [3]

Q8	Solution
(b)(i)	$y = \mathrm{h}(x)$ 
(b)(ii)	$y = \mathrm{h}(\lvert x \rvert)$ 

(b) (iii)	$y = \dfrac{1}{\mathrm{h}(x)}$ 

9 At a robotic exhibition, a miniature robotic mouse is programmed to walk 2 different simulation paths from a starting point O to and from a series of points, $P_1, P_2, P_3, \ldots,$ increasingly far away in a straight line. The robotic mouse starts at O and walks stage 1 from O to P_1 and back to O, then stage 2 from O to P_2 and back to O, and so on. The miniature robotic mouse can walk a maximum distance of 100 m before its battery runs out.

(i)

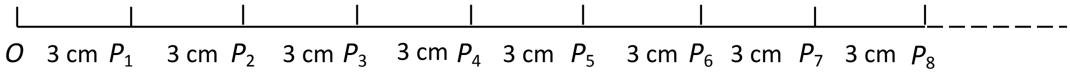


Fig. 1

In Program A, the distances between adjacent points are all 3 cm (see Fig. 1).

(a) Find the distance travelled by the robotic mouse that completes the first 12 stages of program A. [2]

(b) Write down an expression for the distance travelled by the robotic mouse that completes n stages of program A. Hence find the greatest number of stages that the robotic mouse can complete before its battery runs out. [4]

Q9	**Solutions**
(i) **(a)**	Stage 1: 6 = 6(1) Stage 2: 12 = 6(2) Stage 3: 18 = 6(3) Stage 4: 24 = 6(4) ….. Stage 12: 6(12) A.P. : 6,12,18………,72,……….,6N $T_1 = a = 6 \qquad d = 6$ Distance travelled from the 1st stage till the 12th stage $= S_{12} = \frac{6}{2}(12)(1+12) = 468$
(i) **(b)**	Stage N: 6n Distance travelled from the 1st stage till the nth stage $= S_n = \frac{6}{2}(n)(1+n) = 3n(n+1)$ $S_n \le 10000$ $3n(n+1) \le 10000$ <u>Method 1:</u>

	$3n^2+3N-10000\le 0$ $\therefore -58.24\le n\le 57.24$ Greatest n = 57. Method 2: (Use table in GC)

(ii)

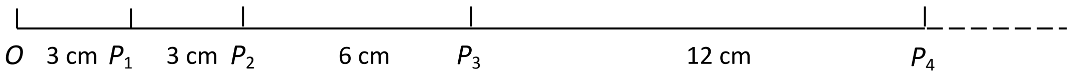


Fig. 2

In program B, the distances between the points are such that OP_1 = 3 cm, P_1P_2 = 3 cm, P_2P_3 = 6 cm and P_nP_{n+1} = $2P_{n-1}P_n$ (see Fig. 2). Write down an expression for the distance travelled by the robotic mouse that completes N stages of program B. Hence find the distance from O, and the direction of travel, of the robotic mouse at the instance when the battery runs out in program B. [5]

Q9	**Solutions**
(ii)	Stage 1: 3+3 = 6(1) Stage 2: 6+6 = 6(2) Stage 3: 12+12 = 6(4) Stage 4: 24+24 = 6(8) ….. Stage N: $6(2^{N-1})$ G.P. $T_1=a=6$ $r=2$ $S_N=\dfrac{6\left(2^N-1\right)}{2-1}$ $S_N=6\left(2^N-1\right)$ Suppose $S_N\le 10000$ $6\left(2^N-1\right)\le 10000$ $\left(2^N-1\right)\le\dfrac{10000}{6}$ $N\le\dfrac{\ln\left(\frac{10006}{6}\right)}{\ln 2}$ or use GC to find $N<10.704$

	Maximum N = 10 for $S_N < 10000$ When n = 10 , $S_{10} = 6138$ Hence remaining distance $= 10000 - 6138$ $= 3862$ cm At stage 11, $U_{11} = 6\left(2^{10}\right) = 6144$ Half of stage 11 is 3072 **Distance from *O*** **= 6144 – 3862 = 2282 cm** The direction of travel is towards O from P_{11} since 3862 > 3072.

10 **(a)** **(i)** Obtain a formula for $\int_{\frac{1}{3}}^{a} \frac{\ln 3x}{x^3}\,\mathrm{d}x$ in terms of a, where $a > \frac{1}{3}$. [4]

(ii) Hence evaluate $\int_{\frac{1}{3}}^{\infty} \frac{\ln 3x}{x^3}\,\mathrm{d}x$. [1]

[You may assume that $\frac{\ln a}{a^2} \to 0$ as $a \to \infty$.]

(b) The region bounded by the curve $y = \frac{3}{1+4x^2}$, the x-axis, the lines $x = -\frac{\sqrt{3}}{2}$ and $x = \frac{\sqrt{3}}{2}$ is rotated through 2π radians about the x-axis. Use the substitution $2x = \tan\theta$ to show that the volume of the solid obtained is given by $9\pi\int_0^{\frac{\pi}{3}} \cos^2\theta\,\mathrm{d}\theta$, and evaluate this integral exactly. [6]

Q10	**Solutions**
(a)(i)	$\int_{\frac{1}{3}}^{a} \frac{\ln 3x}{x^3}\,\mathrm{d}x = \left[(\ln 3x)\left(-\frac{1}{2x^2}\right)\right]_{\frac{1}{3}}^{a} - \int_{\frac{1}{3}}^{a}\left(\frac{3}{3x}\right)\left(-\frac{1}{2x^2}\right)\mathrm{d}x$ $= -\frac{\ln 3a}{2a^2} + \int_{\frac{1}{3}}^{a} \frac{1}{2x^3}\,\mathrm{d}x$ $= -\frac{\ln 3a}{2a^2} + \left[-\frac{1}{4x^2}\right]_{\frac{1}{3}}^{a}$ $= \frac{9}{4} - \frac{\ln 3a}{2a^2} - \frac{1}{4a^2}$
(a)(ii)	As $a \to \infty$, $\frac{1}{4a^2} \to 0$ and $\frac{\ln 3a}{2a^2} = \frac{\ln 3}{2a^2} + \frac{\ln a}{2a^2} \to 0$ Hence, $\int_{\frac{1}{3}}^{\infty} \frac{\ln 3x}{x^3}\,\mathrm{d}x = \frac{9}{4}$
(b)	$2x = \tan\theta \;\Rightarrow\; \mathrm{d}x = \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta$ When $x = 0$, $0 = \tan\theta \;\Rightarrow\; \theta = 0$ When $x = \frac{\sqrt{3}}{2}$, $\sqrt{3} = \tan\theta \;\Rightarrow\; \theta = \frac{\pi}{3}$

Volume of revolution

$$= 2 \times \pi \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{3}{1+4x^2} \right)^2 \, \mathrm{d}x$$

$$= 18\pi \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\left(1+4x^2\right)^2} \, \mathrm{d}x$$

$$= 18\pi \int_0^{\frac{\pi}{3}} \frac{1}{\left(1+\tan^2\right)^2} \cdot \frac{1}{2} \sec^2\theta \, \mathrm{d}\theta$$

$$= 9\pi \int_0^{\frac{\pi}{3}} \frac{1}{\sec^2\theta} \, \mathrm{d}\theta$$

$$= 9\pi \int_0^{\frac{\pi}{3}} \cos^2\theta \, \mathrm{d}\theta \qquad \text{[Shown]}$$

$$9\pi \int_0^{\frac{\pi}{3}} \cos^2\theta \, \mathrm{d}\theta$$

$$= 9\pi \int_0^{\frac{\pi}{3}} \frac{\cos 2\theta + 1}{2} \, \mathrm{d}\theta$$

$$= \frac{9\pi}{2} \left[\frac{\sin 2\theta}{2} + \theta \right]_0^{\frac{\pi}{3}}$$

$$= \frac{9\pi}{2} \left(\frac{\sqrt{3}}{4} + \frac{\pi}{3} \right)$$

11 The plane p_1 has equation $x-2y+3z=5$. The plane p_2 contains the points A and B with coordinates (1, –3, 1) and (4, 3, –2) respectively and is perpendicular to p_1.

(i) Find the cartesian equation of p_2. [3]

(ii) Find the exact perpendicular distance from A to p_1. [2]

(iii) Find the vector equation of the line l where p_1 and p_2 meet. [2]

(iv) Given that C is a general point on l, find an expression for the square of the distance BC. Hence, or otherwise, find the position vector of the point on l which is nearest to B. [5]

Q11	Solutions
(i)	$\overrightarrow{AB}=\begin{pmatrix}4\\3\\-2\end{pmatrix}-\begin{pmatrix}1\\-3\\1\end{pmatrix}=\begin{pmatrix}3\\6\\-3\end{pmatrix}=3\begin{pmatrix}1\\2\\-1\end{pmatrix}$ $\mathbf{n}_2=\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}4\\-4\\-4\end{pmatrix}=4\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ Equation of p_2 $\mathbf{r}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}1\\-3\\1\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ $\mathbf{r}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=3$ $\therefore$ Cartesian equation of p_2: $x-y-z=3$
(ii)	By inspection, the point C with coordinates (5,0,0) lies in plane p_1. $\overrightarrow{AC}=\begin{pmatrix}4\\3\\-1\end{pmatrix}$ Distance from A to p_1 $=\dfrac{\left\lvert\overrightarrow{AC}\cdot\mathbf{n}_1\right\rvert}{\lvert\mathbf{n}_1\rvert}=\dfrac{\left\lvert\begin{pmatrix}4\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\3\end{pmatrix}\right\rvert}{\left\lvert\sqrt{14}\right\rvert}=\dfrac{5}{\sqrt{14}}$

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<u>Method 2</u>
Using distance between plane or using formula:
Distance from A to p_1

$$= \frac{\left|\mathbf{a}\cdot\mathbf{n}_1 - D_1\right|}{\left|\mathbf{n}_1\right|} = \frac{\left|\begin{pmatrix}1\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\3\end{pmatrix}-5\right|}{\left|\sqrt{14}\right|} = \frac{5}{\sqrt{14}}$$

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<tr><td>(iii)</td><td>

Using GC, equation of line l is

$$\mathbf{r} = \begin{pmatrix}1\\-2\\0\end{pmatrix} + \lambda\begin{pmatrix}5\\4\\1\end{pmatrix}, \quad \lambda \in \mathbb{R}$$

OR
Method 2:
Using cross product of the 2 normals $(\mathbf{n}_1 \times \mathbf{n}_2)$ to find the directional vector of the line and then identify a common point to get the equation of the line.

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Since point C is lying on the line l,

$$\overrightarrow{OC} = \begin{pmatrix}1+5\lambda\\-2+4\lambda\\\lambda\end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix}1+5\lambda\\-2+4\lambda\\\lambda\end{pmatrix} - \begin{pmatrix}4\\3\\-2\end{pmatrix}$$

$$= \begin{pmatrix}-3+5\lambda\\-5+4\lambda\\2+\lambda\end{pmatrix}$$

$\left|\overrightarrow{BC}\right|^2$

$= \left(-3+5\lambda\right)^2 + \left(-5+4\lambda\right)^2 + \left(2+\lambda\right)^2$
$= 9-30\lambda+25\lambda^2+25-40\lambda+16\lambda^2+4+4\lambda+\lambda^2$
$= 42\lambda^2-66\lambda+38$

<u>Method 1:</u>
For distance BC to be shortest,

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	$\left\|\overrightarrow{BC}\right\|^2 = 42\lambda^2 - 66\lambda + 38$ $= 42\left[\lambda^2 - \frac{11}{7}\lambda + \left(-\frac{11}{14}\right)^2 - \left(-\frac{11}{14}\right)^2\right] + 38$ $= 42\left(\lambda - \frac{11}{14}\right)^2 + \frac{169}{14}$ $\left\|\overrightarrow{BC}\right\|^2$ is minimum when $\lambda = \frac{11}{14}$ $\therefore \left\|\overrightarrow{BC}\right\|$ is minimum when $\lambda = \frac{11}{14}$ $\overrightarrow{OC} = \begin{pmatrix} 1+5\left(\frac{11}{14}\right) \\ -2+4\left(\frac{11}{14}\right) \\ \frac{11}{14} \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 69 \\ 16 \\ 11 \end{pmatrix}$ <u>Method 2:</u> For distance *BC* to be shortest, let $y = \left\|\overrightarrow{BC}\right\|^2$. $\frac{\mathrm{d}y}{\mathrm{d}\lambda} = 84\lambda - 66 = 0$ $\lambda = \frac{11}{14}$ Since the coefficient of λ^2 is positive, $y = \left\|\overrightarrow{BC}\right\|^2$ is minimum when $\lambda = \frac{11}{14}$. $\overrightarrow{OC} = \begin{pmatrix} 1+5\left(\frac{11}{14}\right) \\ -2+4\left(\frac{11}{14}\right) \\ \frac{11}{14} \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 69 \\ 16 \\ 11 \end{pmatrix}$ Otherwise Method: Since $\overrightarrow{BC}$ is perpendicular to *l*,

$$\begin{pmatrix} -3+5\lambda \\ -5+4\lambda \\ 2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} = 0$$

$$-15+25\lambda-20+16\lambda+2+\lambda=0$$

$$42\lambda=33$$

$$\lambda=\frac{11}{14}$$

$$\overrightarrow{OC}=\begin{pmatrix} 1+5\left(\frac{11}{14}\right) \\ -2+4\left(\frac{11}{14}\right) \\ \frac{11}{14} \end{pmatrix}=\frac{1}{14}\begin{pmatrix} 69 \\ 16 \\ 11 \end{pmatrix}$$