

CANBERRA SECONDARY SCHOOL



# **2024 Preliminary Examination**

## Secondary Four Express

### ADDITIONAL MATHEMATICS

4049/02

26 Aug 2024 2 hours 15 minutes 1130h — 1345h

Name:		
	•	

Class: \_\_\_\_\_

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### READ THESE INSTRUCTIONS FIRST

Write your full name, class and index number on all work you hand in.Write in dark blue or black pen.You may use a pencil for any diagrams or graphs.Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer all the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question. The use of an approved scientific calculator is expected, where appropriate. You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [] at the end of each question or part question.

The total number of marks for this paper is 90.

FOR MARKER'S USE				
	Marks	Max		
	Awarded	Marks		
Total		90		

This question paper consists of <u>22</u> printed pages including the cover page.

Setter: Mr Muhamad Lathif

#### Mathematical Formulae

#### 1. ALGEBRA

Quadratic Equation

For the equation 
$$ax^2 + bx + c = 0, a \neq 0$$
,  

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Binomial expansion

$$(a+b)^{n} = a^{n} + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^{2} + \ldots + \binom{n}{r}a^{n-r}b^{r} + \ldots + b^{n},$$

where *n* is a positive integer and  $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)...(n-r+1)}{r!}$ .

#### 2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\csc^2 A = 1 + \cot^2 A$$
$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for 
$$\triangle ABC$$
  
 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ .  
 $a^2 = b^2 + c^2 - 2bc \cos A$ .  
 $\Delta = \frac{1}{2} ab \sin C$ 

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## Answer all the questions

A curve has an equation 
$$y = 2x\sqrt{2x+1}$$
.  
(a) Show that  $\frac{dy}{dx} = \frac{ax+b}{\sqrt{2x+1}}$ , where *a* and *b* are positive integers. [2]

$$u = 2x$$
  $v = (2x+1)^{\frac{1}{2}} = \sqrt{2x+1}$ 

1

$$\frac{du}{dx} = 2 \qquad \qquad \frac{dv}{dx} = \frac{1}{2} \left( 2x + 1 \right)^{-\frac{1}{2}} (2) = \frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{2x+1}} + 2\sqrt{2x+1}$$
 M1  
$$= \frac{2x+4x+2}{\sqrt{2x+1}} = \frac{6x+2}{\sqrt{2x+1}}$$
 A1

(b) Hence, find 
$$\int \frac{3x+1}{\sqrt{2x+1}} dx$$
. [2]  
 $\int \frac{6x+2}{\sqrt{2x+1}} dx = 2x\sqrt{2x+1} + c$  M1  
 $2\int \frac{3x+1}{\sqrt{2x+1}} dx = 2x\sqrt{2x+1} + c$  A1

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2 A calculator must not be used in this question.

(a) Show that 
$$\sin 15^{\circ} = \frac{\sqrt{6} - \sqrt{2}}{4}$$
. [4]

$$\sin 15^{\circ} = \sin \left(45^{\circ} - 30^{\circ}\right)$$
$$= \sin 45^{\circ} \cos 30^{\circ} - \cos 45^{\circ} \sin 30^{\circ}$$
M1

$$=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)$$
M1

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
M1
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$
A1

(b) Hence, find  $\sin^2 15^\circ$ , giving your answer in the form  $\frac{a-\sqrt{3}}{b}$ , where *a* and *b* are positive integers. [2]

$$\sin^{2} 15^{\circ} = \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)^{2}$$
M1
$$= \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right) \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)$$
$$= \frac{6 - 2\sqrt{12} + 2}{16}$$
$$= \frac{8 - 2\sqrt{12}}{16}$$
$$= \frac{2 - \sqrt{3}}{4}$$
A1

(c) By using part (b), show that  $\cos^2 15^\circ = \frac{2+\sqrt{3}}{4}$ . [2]

$$\sin^{2} 15^{\circ} + \cos^{2} 15^{\circ} = 1$$

$$\frac{2 - \sqrt{3}}{4} + \cos^{2} 15^{\circ} = 1$$

$$\cos^{2} 15^{\circ} = 1 - \frac{2 - \sqrt{3}}{4}$$

$$= \frac{4}{4} - \frac{2 - \sqrt{3}}{4} = \frac{2 + \sqrt{3}}{4}$$
A1

3 (a) Prove that  $\cos ec 2x + \cot 2x = \cot x$ . [4]

6

$$LHS = \cos ec 2x + \cot 2x$$

$$= \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x}$$

$$M1$$

$$= \frac{1 + \cos 2x}{\sin 2x}$$

$$= \frac{1 + 2\cos^2 x - 1}{2\sin x \cos x}$$

$$M1$$

$$= \frac{\cos x}{\sin x}$$

$$M1$$

$$= \cot x = RHS (Proved)$$

(b) Hence, solve  $\cos ecx + \cot x = \sqrt{3}$  where  $0 \le x \le 2\pi$ . [3]

$$\cos ecx + \cot x = \sqrt{3}$$

$$\cot \frac{1}{2}x = \sqrt{3}$$

$$\tan \frac{1}{2}x = \frac{1}{\sqrt{3}}$$
Basic angle = 
$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$
M1

Tangent is positive, 1<sup>st</sup> and 3rd quadrant.

$$\frac{1}{2}x = \frac{\pi}{6}, \frac{7\pi}{6}$$
$$x = \frac{\pi}{3}$$
A1

(a) Solve the equation  $3^x - 3^{2-x} = 8$ . 4 [4]

$$3^{x} - 3^{2-x} = 8$$

$$3^{x} - \frac{3^{2}}{3^{x}} = 8$$
M1
$$3^{2x} - 8(3^{x}) - 9 = 0$$
Let
$$3^{x} = X$$

$$X^{2} - 8X - 9 = 0$$

$$(X + 1)(X - 9) = 0$$
M1
$$3^{x} = -1 (NA) \text{ or } 3^{x} = 9$$
M1
$$x = 2$$
A1

(b) The curve 
$$y = \log_5(2x+5)$$
 intersects the x-axis at A and the y-axis at B.

(i) Find the coordinates of 
$$A$$
 and  $B$ . [3]

Cuts *x*-axis, *y*-coordinate 
$$= 0$$

$$log_{5}(2x+5) = 0$$

$$2x+5=1$$

$$2x = -4$$

$$x = -2$$

$$A(-2, 0)$$
A1

$$x = -2$$
  $A(-2, 0)$ 

Cuts *y*-axis, *x*-coordinate = 0

$$y = \log_5 (2(0) + 5)$$
  
 $y = \log_5 5 = 1$   $B(0, 1)$  B1

(ii) Explain why the graph does not exist for all values of 
$$x \le -\frac{5}{2}$$
. [2]  
 $y = \log_5(2x+5)$  does not exist for  $2x+5 \le 0$  M1

$$2x \le -5$$
$$x \le -\frac{5}{2}$$

A1

5

A compound produced in the laboratory has a growth equation given by [Turn Over

$$w = \frac{600}{10 + 30e^{-0.5t}}$$

where w is the mass in grams and t is the time in hours after the compound was first produced.

Find

(ai) the initial mass of the compound,

[2]

$$w = \frac{600}{10 + 30e^{-0.5t}}$$

Initial mass when t = 0 M1

$$w = \frac{600}{10 + 30e^{-0.5(0)}} = \frac{600}{10 + 30} = 15g$$
 A1

(aii) the mass of the compound after 5 hours 15 minutes, [2]

5h 15 min = 5.25 h M1

$$w = \frac{600}{10 + 30e^{-0.5(5.25)}} = 49.3g$$
 A1

Canberra Secondary School 2024 Preliminary Examination (aiii) the time when the mass is 30g.

$$30 = \frac{600}{10 + 30e^{-0.5t}}$$

$$30(10 + 30e^{-0.5t}) = 600$$

$$10 + 30e^{-0.5t} = 20$$

$$30e^{-0.5t} = 10$$

$$e^{-0.5t} = \frac{1}{3}$$

$$t = -2\ln\left(\frac{1}{3}\right) = 2.20 \text{ hours or } 2 \text{ h } 12 \text{ minutes}$$
A1

[3]

M1

(b) Explain if the mass will continue increasing indefinitely or if it will stabilise to a final mass. [2]

$$t \to \infty,$$
  

$$30e^{-0.5t} \to 0$$

$$M1$$

$$\therefore \frac{600}{10 + 30e^{-0.5t}} \to \frac{600}{10} = 60g$$

Therefore, the compound will stabilize to a final mass of 60g. A1

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- 6 The polynomial  $f(x) = Ax^3 (2A+B)x^2 + 2x + B$ , where A and B are constants, is exactly divisible by (x-1). When f(x) is divided by x, the remainder obtained is 3.
  - (a) Find A and B. [3]

$$f(x) = Ax^{3} - (2A + B)x^{2} + 2x + B$$
  

$$f(0) = A(0)^{3} - (2A + B)(0)^{2} + 2(0) + B = 3$$
  
Therefore  $B = 3$   
A1

$$f(x) = Ax^{3} - (2A+3)x^{2} + 2x + 3$$
  

$$f(1) = A(1)^{3} - (2A+3)(1)^{2} + 2(1) + 3 = 0$$
  

$$A - (2A+3) + 2 + 3 = 0$$
  

$$A - 2A - 3 + 2 + 3 = 0$$
  

$$-A = -2$$
  

$$A = 2$$
  
A1

(b) Factorise f(x) completely.

f (x) = 
$$2x^3 - 7x^2 + 2x + 3$$
  
By long division or solving  $2x^3 - 7x^2 + 2x + 3 = (Ax^2 + Bx + C)(x-1)$   
A = 2  
C = -3  
B1  
B2 = -5  
B1  
 $2x^3 - 7x^2 + 2x + 3 = (2x^2 - 5x - 3)(x-1)$ 

$$=(2x+1)(x-3)(x-1)$$
 B1

[4]

### (c) Hence, or otherwise factorise

$$g(x) = A(x-2)^{3} - (2A+B)(x-2)^{2} + 2(x-2) + B.$$

$$g(x) = A(x-2)^{3} - (2A+B)(x-2)^{2} + 2(x-2) + B$$
M1
$$= 2(x-2)^{3} - 7(x-2)^{2} + 2(x-2) + 3$$

$$= (2(x-2)+1)(x-2-3)(x-2-1)$$
M1
$$= (2x-4+1)(x-5)(x-3)$$

$$= (2x-3)(x-5)(x-3)$$
A1

7 The diagram below shows two horizontal lines *AB* and *CD*, where *L* is the vertical distance between the lines *AB* and *CD*. [Turn Over A rectangle *PQRS* touches lines *AB* and *CD* at *P* and *R* respectively. It is given that PQ = 6 cm, SP = 8 cm and  $\angle QRD = \theta^\circ$ .



Show that  $L = 8\sin\theta + 6\cos\theta$ .

[3]

Draw a vertical line QX from Q to line AB,

$$\angle XQP = \theta \qquad M1$$

$$\cos \theta = \frac{XQ}{6}$$

$$XQ = 6\cos \theta \qquad A1$$

Draw a vertical line from *QY* to line *CD*.

$$\sin \theta = \frac{QY}{6}$$

$$QY = 6\sin \theta$$

$$L = QY + XQ = 8\sin \theta + 6\cos \theta$$
A1

(b) Express  $8\sin\theta + 6\cos\theta$  in the form  $R\sin(\theta + \alpha)$ , where R > 0 and  $0^{\circ} < \alpha < 90^{\circ}$ . Give your answer for  $\alpha$  correct to 2 decimal places. [3]  $8\sin\theta + 6\cos\theta = R\sin(Q + \alpha)$ 

$$R\cos\alpha = 8$$
  $R\sin\alpha = 6$  M1

$$R = \sqrt{8^2 + 6^2} = 10$$
 A1

$$\alpha = \tan^{-1} \frac{6}{8} = 36.9^{\circ}$$
 A1

$$8\sin\theta + 6\cos\theta = 10\sin\left(\theta + 36.9^{\circ}\right)$$

Given that the angle  $\theta$  can vary from  $0^\circ \le \theta < 90^\circ$ 

(c) Hence, state the maximum value of L and the corresponding value(s) of  $\theta$ . [2]

Maximum value of 
$$L = 10 \text{ cm}$$
 B1  
When  $\sin(\theta + 36.9^{\circ}) = 1$   
 $\theta + 36.9^{\circ} = 90^{\circ}$   
 $\theta = 90^{\circ} - 36.9^{\circ} = 53.1^{\circ}$  B1

(d) State the minimum value of L and the corresponding value(s) of 
$$\theta$$
. [2]  
Minimum value of  $L = 6$  cm when  $\theta = 0^{\circ}$  B2

Canberra Secondary School 2024 Preliminary Examination 8 A capsule is made of a cylinder with a hemisphere at one end of the cylinder.
The cylinder has a height, *h* cm and radius, *r* cm.
The hemisphere also has radius, *r* cm.

(Volume of sphere =  $\frac{4}{3}\pi r^3$ , Surface Area of sphere =  $4\pi r^2$ , Volume of cylinder =  $\pi r^2 h$ , Curved surface area of cylinder =  $2\pi rh$ )



Given that the volume of the capsule is  $100 \pi cm^2$ .

(a) Show that 
$$h = \frac{100}{r^2} - \frac{2}{3}r$$
. [2]

Volume of capsule = 
$$\pi r^2 h + \frac{2}{3}\pi r^3 = 100\pi$$
 M1  
 $r^2 h + \frac{2}{3}r^3 = 100$   
 $h + \frac{2}{3}r = \frac{100}{r^2}$   
 $h = \frac{100}{r^2} - \frac{2}{3}r$  A1

(b) Show that the total surface area of the capsule is  $A = \frac{5\pi}{3}r^2 + \frac{200\pi}{r}$ . [2]

Total surface area of capsule,

$$A = 2\pi r^{2} + 2\pi rh + \pi r^{2}$$
  
=  $3\pi r^{2} + 2\pi r \left(\frac{100}{r^{2}} - \frac{2}{3}r\right)$  M1  
=  $3\pi r^{2} + \frac{200\pi}{r} - \frac{4\pi}{3}r^{2}$   
=  $\frac{5\pi}{3}r^{2} + \frac{200\pi}{r}$  A1

 $A = \frac{5\pi}{3}r^{2} + \frac{200\pi}{r}$  $\frac{dA}{dr} = \frac{10\pi}{3}r - \frac{200\pi}{r^{2}} = 0$ M1,M1 $\frac{10\pi}{3}r = \frac{200\pi}{r^{2}}$  $r^{3} = 60$  $r = \sqrt[3]{60} = 3.91cm$ A1

$$\frac{d^2A}{dr^2} = \frac{10\pi}{3} + \frac{400\pi}{r^3}$$
 M1

When  $r^3 = 60$ 

$$\frac{d^2 A}{dr^2} = \frac{10\pi}{3} + \frac{400\pi}{60} > 0$$
 A1

Since 
$$\frac{d^2A}{dr^2} > 0$$
, Area is minimum (By Second Derivative Test). A1

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Additional Mathematics 4049/02 Secondary 4 Express 9 The equation of a circle  $C_1$  is given by the equation  $x^2 + y^2 - 4x - 8y$  **Furn Over** 

Find the coordinates of the center and radius of 
$$C_1$$
. [3]  
 $x^2 + y^2 - 4x - 8y = 0.$   
 $(x-2)^2 + (y-4)^2 - 2^2 - 4^2 = 0$   
 $(x-2)^2 + (y-4)^2 = 20$  M1

Center (2, 4) radius = 
$$\sqrt{20}$$
 or  $2\sqrt{5}$  A1, A1

Point A(0,8) lies on the circle  $C_{1.}$ 

**(a)** 

[3]

(b) Find the equation of tangent to the circle,  $C_1$  at A.

$$M_{OA} = \frac{8-4}{0-2} = -2$$
 M1

Perpendicular gradient =  $\frac{1}{2}$ 

$$8 = \frac{1}{2}(0) + c$$

$$c = 8$$

$$y = \frac{1}{2}x + 8$$
M1
A1

The circle,  $C_1$  is reflected about the y-axis.

(c) Find the equation that represents the reflected circle 
$$C_2$$
. [2]

$$(x+2)^{2} + (y-4)^{2} = 20$$
  
Or  $x^{2} + y^{2} + 4x - 8y = 0.$  A1

A circle,  $C_3$  has the equation  $(x-5)^2 + (y-8)^2 = 5$ 

(d) Explain if 
$$C_1$$
 and  $C_3$  intersect or do not intersect each other. [4]  
Center  $C_3$  (5, 8) radius =  $\sqrt{5}$  M1

Distance between 
$$C_1$$
 and  $C_3 = \sqrt{(5-2)^2 + (8-4)^2} = \sqrt{25} = 5$  units M1

$$R_1 + R_3 = 2\sqrt{5} + \sqrt{5} = 3\sqrt{5}$$
 M1

Since  $R_1 + R_3 >$  Distance between  $C_1$  and  $C_3$ , therefore the 2 circles intersect. A1 10 The diagram shows part of the curve  $y = \frac{15}{(0.5x+1)}$  and a straight line Tyrn Oser-10.

The straight line intersects the *y*-axis at *P*, the *x*-axis at *Q* and the curve at *R*.





$$0 = 1.25x - 10$$
  

$$1.25x = 10$$
  

$$x = \frac{10}{1.25} = 8$$
  

$$Q(8, 0)$$
  
B1

(b) Find the coordinates of 
$$R$$
. [4]

$$\frac{15}{(0.5x+1)} = 1.25x - 10$$
 M1

$$(1.25x-10)(0.5x+1)=15$$
 M1

$$\frac{5x^{2}}{8} - \frac{15x}{4} - 10 = 15$$

$$\frac{5x^{2}}{8} - \frac{15x}{4} - 25 = 0$$

$$5x^{2} - 30x - 200 = 0$$

$$x^{2} - 6x - 40 = 0$$
(x+4)(x-10) = 0
(x = -4 (rej) x = 10)  
y = 1.25(10) - 10 = 2.5
$$R(10, 2.5)$$
A1

(c) Find the area of the shaded region bounded by the curve, the line y=1.25x-10 and the y-axis. [5]

Area of triangle 
$$OPQ = \frac{1}{2} \times 8 \times 10 = 40 units^2$$
 M1

Area of QR = 
$$\frac{1}{2} \times 2 \times 2.5 = 2.5 \text{ units}^2$$
 M1

Area under curve 
$$= \int_0^{10} \frac{15}{(0.5x+1)} dx$$
 M1

$$= [30\ln(0.5x+1)]_{0}^{10}$$
$$= [30\ln 6] - [30\ln 1] = 30\ln 6$$
M1

Shaded area = 
$$30 \ln 6 + 40 - 2.5 = 30 \ln 6 + 37.5 = 91.3 \text{ units}^2$$
 A1

- End of paper -

Additional Mathematics 4049/02 Secondary 4 Express