TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME	
CIVICS GROUP	

H2 MATHEMATICS **9758/01**

Paper 1

18 September 2019

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiners' Use	
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Total	

[Turn Over

1 Using an algebraic method, solve the inequality $\dfrac{4x^2-x-1}{(2x-1)(x+1)} \geq 1$. [4]

Hence or otherwise, solve the inequality

$$\frac{4e^{2x}-e^x-1}{(2e^x-1)(e^x+1)} \geq 1,$$

leaving your answers in exact form. [2]

2 Referred to the origin O, A is a fixed point with position vector $\mathbf{a}$, and $\mathbf{d}$ is a non-zero vector. Given that a general point R has position vector $\mathbf{r}$ such that $\mathbf{r} \times \mathbf{d} = \mathbf{a} \times \mathbf{d}$, show that $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$, where λ is a real constant. Hence give a geometrical interpretation of $\mathbf{r}$. [3]

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$. By writing $\mathbf{r}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, use $\mathbf{r} \times \mathbf{d} = \mathbf{a} \times \mathbf{d}$ to form three equations which represent cartesian equations of three planes. State the relationship between these three planes. [3]

3 **(i)** The sum of the first n terms of a sequence is denoted by S_n. It is known that $S_5 = -30$, $S_{14} = 168$ and $S_{18} = 9S_{10}$. Given that S_n is a quadratic polynomial in n, find S_n in terms of n. [4]

(ii) The nth term of the sequence is denoted by T_n. Find an expression for T_n in terms of n. Hence find the set of values of n for which $|T_n| < 12$. [4]

4 **(i)** Given that f is a strictly increasing continuous function, explain, with the aid of a sketch, why

$$\frac{1}{n}\left\{ f\left(\frac{0}{n}\right) + f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{n-1}{n}\right) \right\} < \int_0^1 f(x)\, dx,$$

where n is a positive integer. [3]

(ii) Hence find the least exact value of k such that $\dfrac{1}{n}\left(e^{\frac{0}{n}} + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}} \right) < k$, where n is a positive integer. [2]

5 It is given that $\mathrm{f}(x)=4x-x^3$.

(i) On separate diagrams, sketch the graphs of $y=|\mathrm{f}(x)|$ and $y=\mathrm{f}(|x|)$, showing clearly the coordinates of any axial intercept(s) and turning point(s). [4]

(ii) Find the exact value of the constant k for which $\int_0^k |\mathrm{f}(x)|\,\mathrm{d}x=\int_{-2}^{2}\mathrm{f}(|x|)\,\mathrm{d}x$. [4]

6 Show that $2\cos(r\theta)\sin\theta \equiv \sin\left[(r+1)\theta\right]-\sin\left[(r-1)\theta\right]$. [1]

By considering the method of differences, find $\sum_{r=1}^{n}\cos(r\theta)$ where $0<\theta<\frac{\pi}{2}$.

(You need not simplify your answer.) [3]

Hence evaluate the sum

$$\cos\left(\frac{19}{6}\pi\right)+\cos\left(\frac{20}{6}\pi\right)+\cos\left(\frac{21}{6}\pi\right)+\cdots+\cos\left(\frac{56}{6}\pi\right)+\cos\left(\frac{57}{6}\pi\right),$$

leaving your answer in exact form. [4]

7 The function f is defined by

$$\mathrm{f}(x)=\begin{cases} n-x & \text{for } n\le x<n+1, \text{where } n \text{ is any positive odd integer,} \\ x-\dfrac{n}{2} & \text{for } n\le x<n+1, \text{where } n \text{ is any positive even integer.} \end{cases}$$

(i) Show that $\mathrm{f}(1.5)=-0.5$ and find $\mathrm{f}(2.5)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $1\le x<5$. [2]

(iii) Does f have an inverse for $1\le x<5$? Justify your answer. [2]

(iv) The function g is defined by $\mathrm{g}:x\mapsto\dfrac{2x-1}{x+1}, x\in\mathbb{R}, x\ne-1$. For $2\le x<3$, find an expression for $\mathrm{gf}(x)$ and hence, or otherwise, find $(\mathrm{gf})^{-1}\left(\dfrac{2}{3}\right)$. [4]

[Turn Over

8 At the start of an experiment, a particular solid substance is placed in a container filled with water. The solid substance will begin to gradually dissolve in the water. Based on experimental data, a student researcher guesses that the mass, x grams, of the remaining solid substance at time t seconds after the start of the experiment satisfies the following differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{k-1}(x-1)(x-k),$$

where k is a real constant and $k>3$.

(i) Show that a general solution of this differential equation is $\ln\left|\frac{x-k}{x-1}\right|=t+C$, where C is an arbitrary real constant. [3]

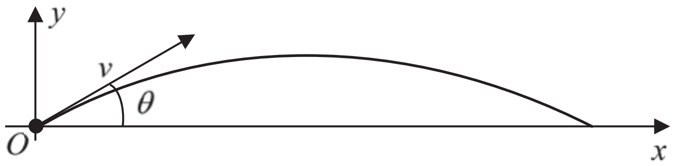
For the rest of the question, let $k=4$. It is given that the initial mass of the solid substance is 3 grams.

(ii) Express the particular solution of the differential equation in the form $x=\mathrm{f}(t)$. [4]

(iii) Find the exact time taken for the mass of the solid substance to become half of its initial value. [2]

(iv) Sketch the part of the curve with the equation found in part **(ii)** which is relevant in this context. [2]

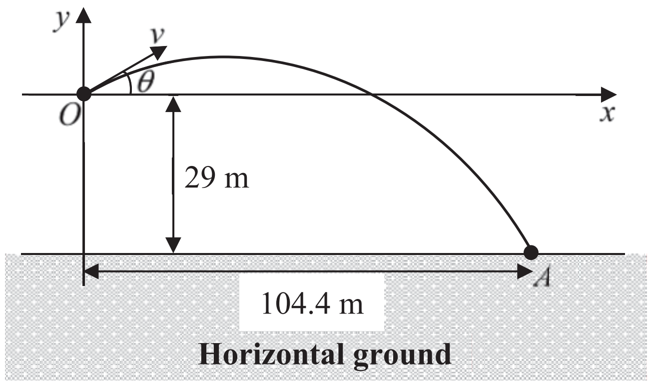
9



From a point O, a particle is projected with velocity $v\ \text{ms}^{-1}$ at a fixed angle of elevation θ from the horizontal, where v is a positive real constant and $0<\theta<\frac{\pi}{2}$. The horizontal displacement, x metres, and the vertical displacement, y metres, of the particle at time t seconds may be modelled by the parametric equations

$$x=(v\cos\theta)t, \qquad y=(v\sin\theta)t-5t^2.$$

(i) Using differentiation, find the maximum height achieved by the particle in terms of v and θ. (You need not show that the height is a maximum.) [3]



The particle is now projected from point O situated at a height of 29 m above the horizontal ground. The particle hits the ground at A which is at a horizontal distance of 104.4 m from O. The maximum height (measured from horizontal ground) that the particle reaches is 57.8 m. The diagram above shows the path of the particle (not drawn to scale).

(ii) Find the time taken for the particle to hit the ground at A and find the corresponding value of v. [5]

(iii) Find the exact gradient of the tangent at A. [2]

[Turn Over

10 Two houses, A and B, have timber cladding on the end of their shed roofs, consisting of rectangular planks of decreasing length.

(i) The first plank of the roof of house A has length 350 cm and the lengths of the planks form a geometric progression. The 20^{th} plank has length 65 cm. Show that the total length of all the planks must be less than 4128 cm, no matter how many planks there are. [4]

House B consists of only 20 planks which are identical to the first 20 planks of house A.

(ii) The total length of all the planks used for house B is L cm. Find the value of L, leaving your answer to the nearest cm. [2]

(iii) Unfortunately the construction company misunderstands the instructions and covers the roof of house B wrongly, so that the lengths of the planks are in arithmetic progression with common difference d cm. If the total length of the 20 planks is still L cm and the length of the 20^{th} plank is still 65 cm, find the value of d and the length of the longest plank. [4]

It is known that house C has timber cladding on the end of its shed roof, consisting of rectangular planks of increasing length. The first plank of the roof of house C has length 65 cm and the lengths of the planks are in arithmetic progression with common difference 11 cm. The total length of the first N planks of the roof of house C exceeds 20 640 cm. Find the least value of N. [3]

11 A circle with a fixed radius r cm is inscribed in an isosceles triangle ABC where $\angle ABC = \theta$ radians and $AB = BC$. The circle is in contact with all three sides of the triangle at the points D, E and F, as shown in Fig. 1.

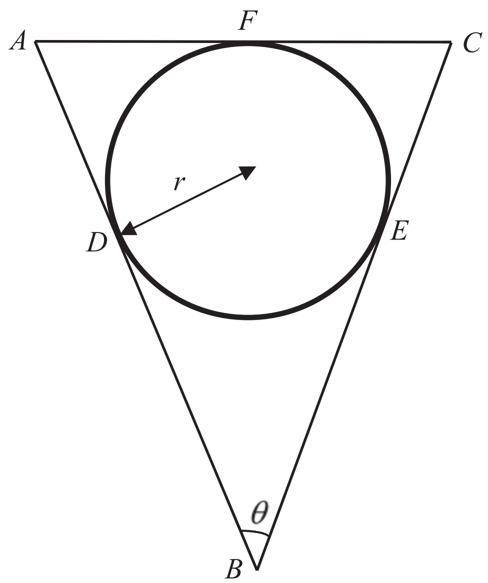


Fig. 1

(i) Show that the length BD can be expressed as $r\cot\dfrac{\theta}{2}$ cm. [1]

(ii) By finding the length AD in terms of r and θ, show that the perimeter of the triangle ABC can be expressed as $4r\cot\left(\dfrac{\pi}{4}-\dfrac{\theta}{4}\right)+2r\cot\dfrac{\theta}{2}$ cm. [2]

(iii) Using differentiation, find the exact value of θ such that the perimeter of the triangle ABC is minimum. Find the minimum perimeter of triangle ABC, leaving your answer in the form $a\sqrt{b}r$ cm, where a and b are positive integers to be determined. [6]

[Turn Over

Fig. 2 shows a decorative item in the shape of a sphere with a fixed radius inscribed in an inverted right circular cone with base radius R cm and slant height $2R$ cm. The sphere is in contact with the slopes and the base of the cone.

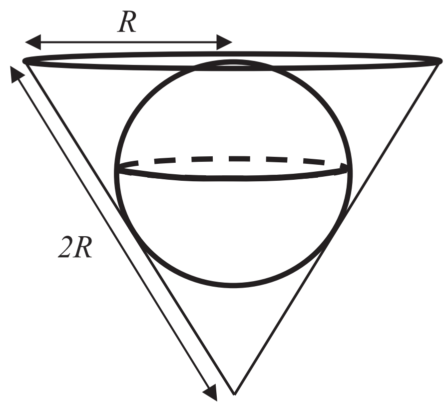


Fig. 2

To make the item glow in the dark, the sphere is filled entirely with fluorescent liquid. However, due to a manufacturing defect, the fluorescent liquid leaks into the bottom of the inverted cone at a rate of 2 cm^3 per minute.

(iv) Assuming that the leaked liquid in the inverted cone will not reach the exterior of the sphere, find the exact rate of increase of the depth of the leaked liquid in the inverted cone when the volume of the leaked liquid in the inverted cone is 24π cm^3. Express your answer in terms of π. [6]

[The volume of a cone of base radius r and height h is given by $\frac{1}{3}\pi r^2 h$.]

End of Paper

TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ____

CIVICS GROUP: ____

H2 MATHEMATICS

9758/02

Paper 2

25 SEPTEMBER 2019

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

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[Turn Over

Section A: Pure Mathematics [40 marks]

1 Given that $y=\sqrt{\left(5-\mathrm{e}^{2x}\right)}$, show that

$$y\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^{2x} \ . \qquad [1]$$

By further differentiation of this result, find the Maclaurin series for y up to and including the term in x^2. [4]

2 Let z_1 and z_2 be the roots of the equation

$$z^2-\mathrm{i}kz-1=0$$

where $0<k<2$ and $0<\arg\left(z_1\right)<\arg\left(z_2\right)<\pi$.

(i) Find z_1 and z_2 in cartesian form, $x+\mathrm{i}y$, where x and y are real constants in terms of k. [3]

For the rest of the question, let $\arg\left(z_1\right)=\theta$, $0<\theta<\frac{\pi}{2}$.

Let w be a complex number such that wz_1 is purely imaginary and $\arg\left(\frac{z_1}{w}\right)=\frac{7\pi}{6}$.

(ii) Show that $\arg\left(z_1\right)=\frac{\pi}{3}$. [3]

(iii) Find z_2, leaving your answer in the exact form. [3]

3 The plane p_1 contains the point A with coordinates $(1,\ 2,\ 8)$ and the line l with equation

$\mathbf{r}=\begin{pmatrix}1\\-4\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-2\end{pmatrix}$, where λ is a real parameter.

(i) Show that a cartesian equation of plane p_1 is $3x-y+z=9$. [2]

The foot of perpendicular from point A to line l is denoted as point F.

(ii) Find the coordinates of point F. [3]

(iii) Point B has coordinates $(1,-4,\ 2)$. Find the exact area of triangle ABF. [2]

(iv) Point C has coordinates $(-1,\ 6,\ 6)$. By finding the shortest distance from point C to p_1, find the exact volume of tetrahedron $ABFC$. [4]

[Volume of tetrahedron = $\frac{1}{3}\times$ base area $\times$ perpendicular height]

(v) Point D lies on the line segment AC such that $AD:DC=1:3$. Another plane p_2 is parallel to p_1 and contains point D. Find a cartesian equation of p_2. [2]

4 A designer is tasked to design a 3-dimensional ornament for the company. He then programs two curves, C_1 and C_2, into the computer software. The curve C_1 has the equation $y=\sqrt{(2-x^2)}$ and the curve C_2 has the equation $y=x^3$. The coordinates of the point of intersection of C_1 and C_2 is $(1,1)$.

(i) Find the exact area of the finite region bounded by C_1, C_2 and the positive x-axis. [You may use the substitution $x=\sqrt{2}\sin\theta$ where $0\le\theta\le\frac{\pi}{2}$.] [6]

(ii) The designer wants to know how much material is needed to construct the 3-dimensional ornament. He finds out that the surface area generated by the segment of a curve $y=\mathrm{f}(x)$ between $x=a$ and $x=b$ rotated through 360° about the x-axis is given by

$$S=\int_a^b 2\pi y\sqrt{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)}\,\mathrm{d}x \text{ where } y=\mathrm{f}(x),\ y\ge 0,\ a\le x\le b.$$

The 3-dimensional ornament is formed when the finite region bounded by C_1, C_2 and the positive x-axis is rotated through 360° about the x-axis. Find the exact surface area of the 3-dimensional ornament. [7]

Section B: Probability and Statistics [60 marks]

5 A biased die in the form of a regular tetrahedron has its four faces labelled 2, 3, 4 and 5, with one number on each face. The die is tossed and X is the random variable denoting the number on the face which the die lands. The probability distribution of X is shown in the table below, where $0<v<u<1$.

x	2	3	4	5
$\mathrm{P}(X=x)$	u	v	u	v

(i) Find $\mathrm{E}(X)$ in terms of u. [2]

(ii) Given that $\mathrm{Var}(X)=1.16$, find u and v. [4]

6 A computer game consists of at most 3 rounds. The game will stop when a player clears 2 rounds or does not clear 2 consecutive rounds. The probability that a player clears round 1 is 0.6. The conditional probability that the player clears round 2 given that he clears round 1 is half the probability that he clears round 1. The conditional probability that the player clears round 2 given that he does not clear round 1 is the same as the probability that he clears round 1.

(i) Find the probability that a player plays 3 rounds. [1]

(ii) Find the probability that a player clears round 1 given that he does not clear round 2. [2]

(iii) The total probability that a player plays 3 rounds and clears round 3 is 0.2. Find the probability that a player clears exactly 2 rounds. [2]

In order to play the computer game, Eric needs to type a 6-digit passcode to unlock the game. The 6-digit passcode consists of digits 0-9 and the digits do not repeat.

How many possible passcodes can there be if

(iv) the 6-digit passcode is odd? [2]

(v) there are exactly 3 odd digits in the 6-digit passcode? [2]

7 MHL bakery sells mini breads that weighs an average of 45 grams each. A customer claims that the bakery is overstating the average weight of mini breads. To test this claim, a random sample of 80 mini breads are selected from MHL bakery and the weight, x grams, of each mini bread is measured. The results are summarised as follows.

$$n = 80 \qquad \sum x = 3571 \qquad \sum x^2 = 159701$$

Calculate unbiased estimates of the population mean and variance of the weight of mini breads. [2]

Test, at the 4% level of significance, whether there is sufficient evidence to support the customer's claim. [4]

From past records, it is known that the weights of the mini breads from MHL bakery are normally distributed with standard deviation 1.5 grams. To further investigate the customer's claim, the bakery records the weights of another 20 randomly selected mini breads and the average weight for the second sample is k grams.
Based on the combined sample of 100 mini breads, find the range of values of k such that the customer's claim is valid at the 4% level of significance. [4]

8 In a chemical reaction, the amount of catalyst used, x grams, and the resulting reaction times, y seconds, were recorded and the results are given in the table.

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
y	62.1	51.2	44.1	39.1	35.0	k	33.0	31.4	29.5

The equation of the regression line of y on x is $y = 68.8067 - 7.12667x$, correct to 6 significant figures.

(i) Show that $k = 37.3$, correct to 1 decimal place. [2]

(ii) Draw a scatter diagram for these values, labelling the axes clearly. [1]

It is suggested that the relationship between x and y can be modelled by one of the following formulae

(A) $y = a + bx$

(B) $y = c + dx^2$

(C) $y = e + \dfrac{f}{x}$

where a, b, c, d, e and f are constants.

(iii) Find the value of the product moment correlation coefficient for each model. Explain which is the best model and find the equation of a suitable regression line for this model. [5]

(iv) By using the equation of the regression line found in part **(iii)**, estimate the reaction time when the amount of catalyst used is 4.2 grams. Comment on the reliability of your estimate. [2]

9 The masses in grams of Envy apples have the distribution $\mathrm{N}\left(\mu,\sigma^2\right)$.

(i) For a random sample of 8 Envy apples, it is given that the probability that the sample mean mass is less than 370 grams is 0.25, and the probability that the total mass of these 8 Envy apples exceeds 3000 grams is 0.5. Find the values of μ and σ. [4]

For the rest of the question, use $\mu = 380$ and $\sigma = 20$.

(ii) Find the probability that the total mass of 8 randomly chosen Envy apples is between 2900 grams and 3100 grams. [2]

The masses in grams of Bravo apples have the distribution $\mathrm{N}\left(250,18^2\right)$.

To make a fruit platter, a machine is used to slice the apples and remove the cores. After slicing and removing the cores, the mass of an Envy apple and the mass of a Bravo apple will be reduced by 30% and 20% respectively. A fruit platter consists of 8 randomly chosen Envy apples and 12 randomly chosen Bravo apples.

(iii) Find the probability that the total mass of fruits, after slicing and removing the cores, in a fruit platter exceeds 4.5 kg. [4]

(iv) State an assumption needed for your calculations in parts **(ii)** and **(iii)**. [1]

To beautify the fruit platter, fruit carving is done on the apples after slicing and removing their cores. The carving reduced the masses of each apple (after slicing and removing its core) by a further 10%.

Let p be the probability that the total mass of fruits in a fruit platter, with carving done, exceeds 4.1 kg. Without calculating p, explain whether p is higher, lower or the same as the answer in part **(iii)**. [1]

10 **(a)** In a packet of 10 sweets, it is given that six of them are red, three of them are yellow and the remaining one is blue. 5 sweets are chosen randomly from the packet of sweets and R denotes the number of sweets that are red. Explain clearly why R cannot be modelled by a binomial distribution. [1]

(b) In a food company, a large number of sweets are produced daily and it is given that $100p\%$ of the sweets produced are red. The sweets are packed into packets of 10 each. Assume now that the number of sweets that are red in a packet follows a binomial distribution.

(i) It is given that the probability of containing exactly five red sweets in a randomly chosen packet of sweets is 0.21253. Show that p satisfies an equation of the form $p(1-p)=k$, where k is a constant to be determined. Hence find the possible values of p. [3]

For the rest of the question, use $p=0.6$.

(ii) A packet of sweets is randomly chosen. Find the probability that there is at most 8 red sweets given that there is more than 2 red sweets. [3]

(iii) Two packets of sweets are selected at random. Find the probability that one of the packets contains at most 5 red sweets and the other packet contains at least 5 red sweets. [3]

(iv) Two packets of sweets are selected at random. Find the probability that the difference in the number of red sweets in the two packets is at least 8. [3]

End of Paper

Qn	Solution	Mark Scheme
1	**Inequalities**	[6]

$$\frac{4x^2-x-1}{(2x-1)(x+1)}\geq 1$$

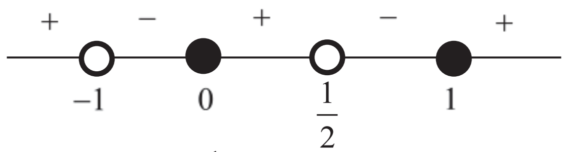
$$\frac{4x^2-x-1}{(2x-1)(x+1)}-1\geq 0$$

$$\frac{(4x^2-x-1)-(2x-1)(x+1)}{(2x-1)(x+1)}\geq 0$$

$$\frac{(4x^2-x-1)-(2x^2+x-1)}{(2x-1)(x+1)}\geq 0$$

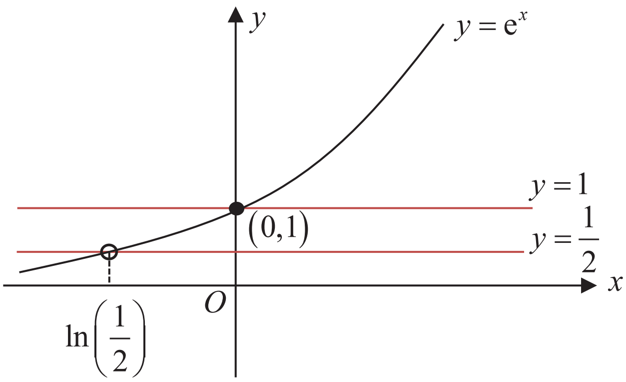
$$\frac{2x^2-2x}{(2x-1)(x+1)}\geq 0$$

$$\frac{2x(x-1)}{(2x-1)(x+1)}\geq 0$$



$x<-1$ or $0\leq x<\frac{1}{2}$ or $x\geq 1$

For $\dfrac{4e^{2x}-e^x-1}{(2e^x-1)(e^x+1)}\geq 1$, replace x by e^x.



From previous result, $e^x<-1$ or $0\leq e^x<\frac{1}{2}$ or $e^x\geq 1$

For $e^x<-1$, there is no real solution.

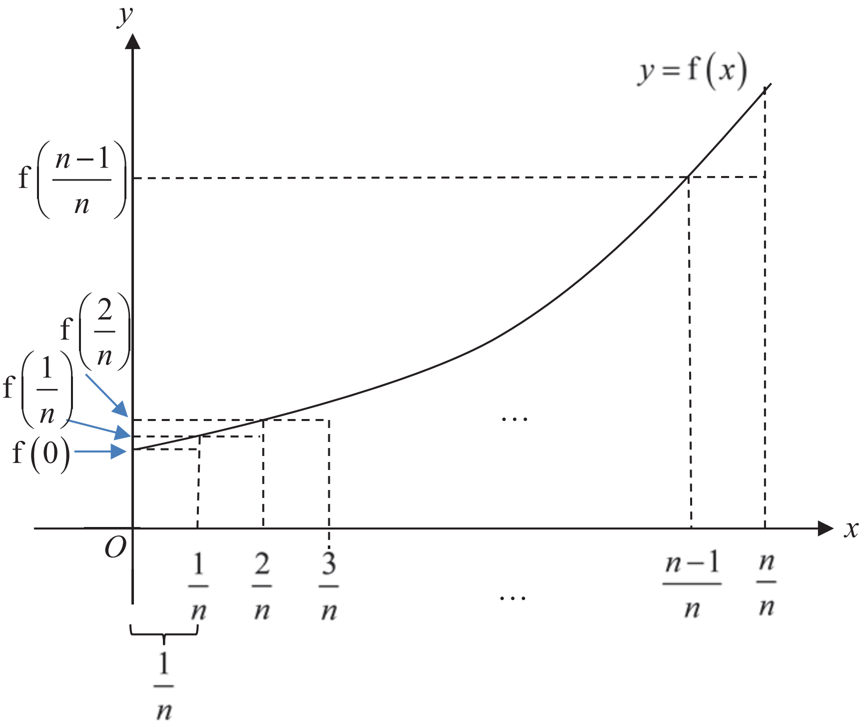
For $0\leq e^x<\frac{1}{2}$ $\Rightarrow x<\ln\frac{1}{2}$ $(\text{or } x<-\ln 2)$

For $e^x\geq 1$, $x\geq 0$

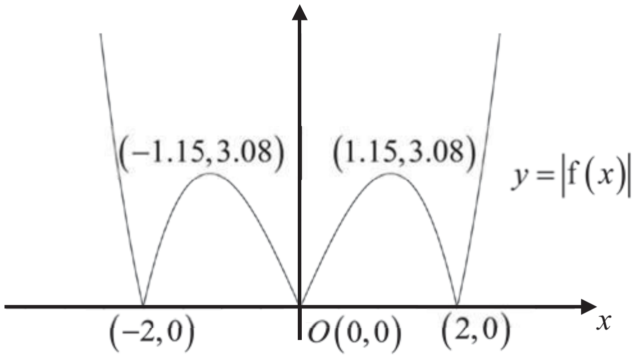
$\therefore x<\ln\frac{1}{2}$ or $x\geq 0$

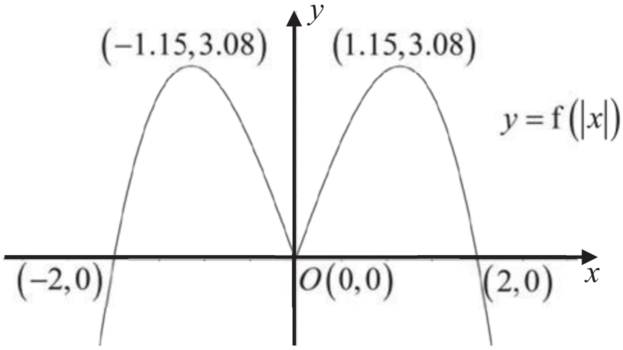
Qn	Solution	Mark Scheme
2	**Vectors**	[6]
	$\mathbf{r}\times\mathbf{d}=\mathbf{a}\times\mathbf{d}$ $\mathbf{r}\times\mathbf{d}-\mathbf{a}\times\mathbf{d}=\mathbf{0}$ $(\mathbf{r}-\mathbf{a})\times\mathbf{d}=\mathbf{0}$ $\because \mathbf{d}\neq\mathbf{0},\ \mathbf{r}-\mathbf{a}=\lambda\mathbf{d},\quad \lambda\in\mathbb{R}$ $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$ **r** is the position vector of a point on the line which passes through point A and parallel to **d**.	
	$\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\1\\5\end{pmatrix}$ $\begin{pmatrix}5y-z\\2z-5x\\x-2y\end{pmatrix}=\begin{pmatrix}7\\1\\-3\end{pmatrix}$ $5y-z=7$ $2z-5x=1$ $x-2y=-3$ These three planes intersect at the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$.	

Qn	Solution	Mark Scheme
3	**System of Linear Equations and Series**	**[8]**
(i)	Let $S_n = an^2 + bn + c$ $S_5 = -30$ $\Rightarrow a(5)^2 + 5b + c = -30$ $\Rightarrow 25a + 5b + c = -30 ---------(1)$ $S_{14} = 168$ $\Rightarrow a(14)^2 + 14b + c = 168$ $\Rightarrow 196a + 14b + c = 168 ---------(2)$ $S_{18} = 9S_{10}$ $\Rightarrow a(18)^2 + 18b + c = 9\left[a(10)^2 + 10b + c\right]$ $\Rightarrow 576a + 72b + 8c = 0 ---------(3)$ Solving, $a = 2, b = -16, c = 0$ $\therefore S_n = 2n^2 - 16n$ Note: if student start with sum of AP formula, then give full credit	
(ii)	$S_n = 2n^2 - 16n$ $S_{n-1} = 2(n-1)^2 - 16(n-1)$ $T_n = S_n - S_{n-1}$ $= 2n^2 - 16n - \left[2(n-1)^2 - 16(n-1)\right]$ $= 2n^2 - 16n - \left(2n^2 - 4n + 2 - 16n + 16\right)$ $= 4n - 18$ $\lvert 4n - 18 \rvert < 12$ Using GC, $1.5 < n < 7.5$ The set of values of n is $\left\{n \in \mathbb{Z}^+ : 2 \le n \le 7\right\}$	

Qn	Solution	Mark Scheme
4	**Applications of Integration**	**[5]**
(i)	 Using integration, area bounded by $y=\mathrm{f}(x)$, $x=0$, $x=1$ and the x-axis is given by $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$. Also, the same area can be estimated by first, dividing x values from 0 to 1 into n equal parts, forming n rectangles of equal width of $\frac{1}{n}$. Hence, the same area is estimated by finding the total areas of the rectangles $=\frac{1}{n}\times\mathrm{f}\left(\frac{0}{n}\right)+\frac{1}{n}\times\mathrm{f}\left(\frac{1}{n}\right)+\frac{1}{n}\times\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\frac{1}{n}\times\mathrm{f}\left(\frac{n-1}{n}\right)$ $=\frac{1}{n}\left\{\mathrm{f}\left(\frac{0}{n}\right)+\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\mathrm{f}\left(\frac{n-1}{n}\right)\right\}$ As the sum of areas of these n rectangles is less than the actual area bounded by $y=\mathrm{f}(x)$, $x=0$, $x=1$ and the x-axis $\frac{1}{n}\left\{\mathrm{f}\left(\frac{0}{n}\right)+\mathrm{f}\left(\frac{1}{n}\right)+\ldots+\mathrm{f}\left(\frac{n-1}{n}\right)\right\}<\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ (shown) [**Note:** If student shows graph that is overestimation/strictly decreasing function, and show the relationship between $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ and area of n rectangles, then give 1M]	

(ii)	$\frac{1}{n}\left(\mathrm{e}^{\frac{0}{n}}+\mathrm{e}^{\frac{2}{n}}+\mathrm{e}^{\frac{4}{n}}+\ldots+\mathrm{e}^{\frac{2n-2}{n}}\right)=\frac{1}{n}\left\{\mathrm{f}\left(\frac{0}{n}\right)+\mathrm{f}\left(\frac{1}{n}\right)+\ldots+\mathrm{f}\left(\frac{n-1}{n}\right)\right\}$, where $\mathrm{f}(x)=\mathrm{e}^{2x}$ $<\int_0^1 \mathrm{e}^{2x}\ \mathrm{d}x$ $=\frac{1}{2}\left[\mathrm{e}^{2x}\right]_0^1$ $=\frac{1}{2}\left(\mathrm{e}^2-1\right)$ $\therefore$ least value of $k=\frac{1}{2}\left(\mathrm{e}^2-1\right)$	

<table>
<tr><th>Qn</th><th>Solution</th><th>Mark Scheme</th></tr>
<tr><td>5</td><td>Graphing Techniques, Transformations and Area under graph</td><td>[8]</td></tr>
<tr><td>(i)</td><td>




</td><td></td></tr>
<tr><td>(ii)</td><td>

$$\int_0^k \left|\mathrm{f}(x)\right| \mathrm{d}x = \int_{-2}^{2} \mathrm{f}(|x|)\ \mathrm{d}x$$

$$\int_0^k \left|\mathrm{f}(x)\right| \mathrm{d}x = 8$$

$$\int_0^2 \left(4x - x^3\right) \mathrm{d}x + \int_2^k -\left(4x - x^3\right) \mathrm{d}x = 8$$

$$4 + \int_2^k -\left(4x - x^3\right) \mathrm{d}x = 8$$

$$\int_2^k \left(x^3 - 4x\right) \mathrm{d}x = 4$$

$$\left[\frac{1}{4}x^4 - 2x^2\right]_2^k = 4$$

$$\left(\frac{1}{4}k^4 - 2k^2\right) - \left(\frac{1}{4}(2)^4 - 2(2)^2\right) = 4$$

$$\frac{1}{4}k^4 - 2k^2 + 4 = 4$$

$$k^4 - 8k^2 = 0$$

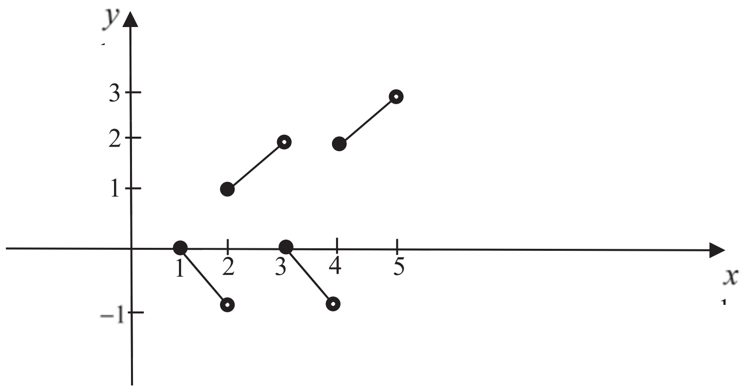
$$k^2 = 8 \qquad (\because k \neq 0)$$

$$k = 2\sqrt{2} \qquad \text{or}$$

$$-2\sqrt{2} \quad (\text{rejected } \because k > 2)$$

</td><td></td></tr>
</table>

Qn	Solution	Mark Scheme
6	**Summation and Method of Differences**	**[8]**
	$2\cos(r\theta)\sin\theta$ $=\sin(r\theta+\theta)-\sin(r\theta-\theta)$ $=\sin\left[(r+1)\theta\right]-\sin\left[(r-1)\theta\right]$ (shown)	
	$\sum_{r=1}^{n}\cos(r\theta)$ $=\frac{1}{2\sin\theta}\sum_{r=1}^{n}\left[\sin\left[(r+1)\theta\right]-\sin\left[(r-1)\theta\right]\right]$ $=\frac{1}{2\sin\theta}[\sin 2\theta-\sin 0$ $+\sin 3\theta-\sin\theta$ $+\sin 4\theta-\sin 2\theta$ $+\sin 5\theta-\sin 3\theta$ $+\ldots$ $+\sin\left[(n-1)\theta\right]-\sin\left[(n-3)\theta\right]$ $+\sin(n\theta)-\sin\left[(n-2)\theta\right]$ $+\sin\left[(n+1)\theta\right]-\sin\left[(n-1)\theta\right]]$ $=\frac{1}{2\sin\theta}\left[\sin(n\theta)+\sin\left[(n+1)\right]\theta-\sin\theta\right]$	
	$\cos\left(\frac{19}{6}\pi\right)+\cos\left(\frac{20}{6}\pi\right)+\cos\left(\frac{21}{6}\pi\right)+\cdots+\cos\left(\frac{56}{6}\pi\right)+\cos\left(\frac{57}{6}\pi\right)$ $=\sum_{r=19}^{57}\cos\left[r\left(\frac{\pi}{6}\right)\right]$ $=\sum_{r=1}^{57}\cos\left[r\left(\frac{\pi}{6}\right)\right]-\sum_{r=1}^{18}\cos\left[r\left(\frac{\pi}{6}\right)\right]$ $=\frac{1}{2\sin\frac{\pi}{6}}\left[\sin\left(\frac{57}{6}\pi\right)+\sin\left(\frac{58}{6}\pi\right)-\sin\frac{\pi}{6}\right]$ $-\frac{1}{2\sin\frac{\pi}{6}}\left[\sin\left(\frac{18}{6}\pi\right)+\sin\left(\frac{19}{6}\pi\right)-\sin\frac{\pi}{6}\right]$ $=\left(-1-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)-\left(0-\frac{1}{2}-\frac{1}{2}\right)$ $=-\frac{1}{2}-\frac{\sqrt{3}}{2}$	

Qn	Solution	Mark Scheme
7	**Functions**	[10]
(i)	For $1 \le x < 2$, $\mathrm{f}(x) = 1 - x$ $\therefore \mathrm{f}(1.5) = 1 - 1.5 = -0.5$ (shown) For $2 \le x < 3$, $\mathrm{f}(x) = x - 1$ $\therefore \mathrm{f}(2.5) = 2.5 - 1 = 1.5$	
(ii)	For $1 \le x < 2$, $\mathrm{f}(x) = 1 - x$ For $2 \le x < 3$, $\mathrm{f}(x) = x - 1$ For $3 \le x < 4$, $\mathrm{f}(x) = 3 - x$ For $4 \le x < 5$, $\mathrm{f}(x) = x - 2$ 	
(iii)	Since $\mathrm{f}(1) = 0 = \mathrm{f}(3)$, f is not a one-to-one function. Hence, f does not have an inverse. **<u>Alternative</u>** Since $y = 0$ cuts the graph of f at 2 distinct points, f is not a one-to-one function. Hence, f does not have an inverse.	
(iv)	For $2 \le x < 3$, $\mathrm{f}(x) = x - 1$ $\mathrm{gf}(x) = \dfrac{2(x-1)-1}{(x-1)+1} = \dfrac{2x-3}{x}$ Let $x = (\mathrm{gf})^{-1}\left(\dfrac{2}{3}\right)$ $\mathrm{gf}(x) = \dfrac{2}{3}$ $\dfrac{2x-3}{x} = \dfrac{2}{3}$ $x = \dfrac{9}{4}$	

	Alternative: For $2 \le x < 3$, $f(x) = x-1$, $gf(x) = \frac{2(x-1)-1}{(x-1)+1} = \frac{2x-3}{x}$ Let $y = \frac{2x-3}{x}$ $xy = 2x-3$ $2x - xy = 3$ $x(2-y) = 3$ $x = \frac{3}{2-y}$ $(gf)^{-1}(x) = \frac{3}{2-x}$ $(gf)^{-1}\left(\frac{2}{3}\right) = \frac{3}{2-\left(\frac{2}{3}\right)} = \frac{9}{4}$	

Qn	Solution	Mark Scheme
8	**Differential Equations**	**[11]**

(i)

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{k-1}(x-1)(x-k)$$

$$\int \frac{1}{(x-1)(x-k)}\ \mathrm{d}x=\int \frac{1}{k-1}\ \mathrm{d}t$$

$$\int \frac{\frac{1}{k-1}}{(x-k)}+\frac{\frac{1}{1-k}}{(x-1)}\ \mathrm{d}x=\int \frac{1}{k-1}\ \mathrm{d}t$$

$$\int \frac{1}{(x-k)}-\frac{1}{(x-1)}\ \mathrm{d}x=\int 1\ \mathrm{d}t$$

$$\ln|x-k|-\ln|x-1|=t+C$$ where C is an arbitrary real constant

$$\ln\left|\frac{x-k}{x-1}\right|=t+C$$

<u>Alternative Method (Completing Square) – to delete before rolling out to students</u>

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{k-1}\left(x^2-(1+k)x+k\right)$$

$$=\frac{1}{k-1}\left(\left(x-\frac{1+k}{2}\right)^2-\left(\frac{1+k}{2}\right)^2+k\right)$$

$$=\frac{1}{k-1}\left(\left(x-\frac{1+k}{2}\right)^2-\frac{1-2k+k^2}{4}\right)$$

$$=\frac{1}{k-1}\left(\left(x-\frac{1+k}{2}\right)^2-\left(\frac{k-1}{2}\right)^2\right)$$

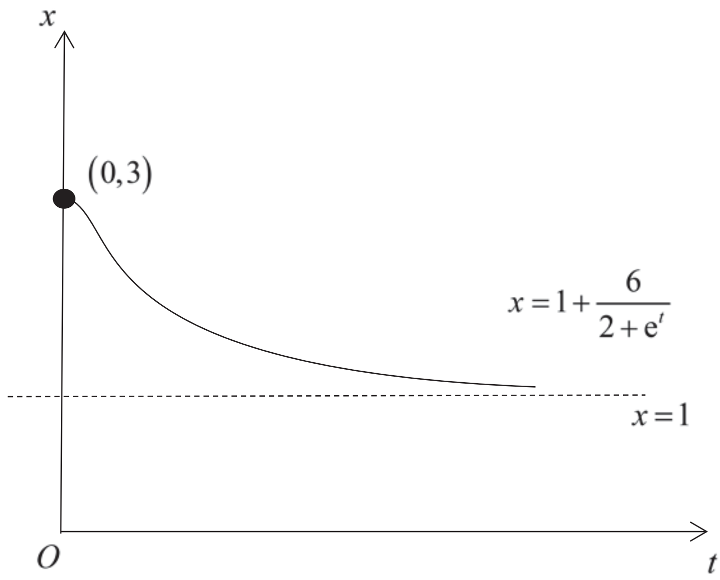
$$\int \frac{1}{\left(x-\frac{1+k}{2}\right)^2-\left(\frac{k-1}{2}\right)^2}\ \mathrm{d}x=\int \frac{1}{k-1}\ \mathrm{d}t$$

$$\frac{1}{2\left(\frac{k-1}{2}\right)}\ln\left|\frac{x-\frac{1+k}{2}-\frac{k-1}{2}}{x-\frac{1+k}{2}+\frac{k-1}{2}}\right|=\frac{1}{k-1}t+D$$ where D is an arbitrary real constant

$$\frac{1}{k-1}\ln\left|\frac{x-\frac{1+k+k-1}{2}}{x-\frac{1+k-k+1}{2}}\right|=\frac{1}{k-1}t+D$$

$$\ln\left|\frac{x-k}{x-1}\right|=t+C\ ,C=(k-1)D$$

(ii)	$\ln\left\|\frac{x-4}{x-1}\right\| = t + C$ $\frac{x-4}{x-1} = Ae^t$ where $A = \pm e^C$ When $t = 0$, $x = 3$, $\frac{3-4}{2} = A$ $A = -\frac{1}{2}$ $\frac{x-4}{x-1} = -\frac{1}{2}e^t$ **<u>Method 1</u>** $1 - \frac{3}{x-1} = -\frac{1}{2}e^t$ $\frac{3}{x-1} = 1 + \frac{1}{2}e^t$ $\frac{x-1}{3} = \frac{1}{1+\frac{1}{2}e^t}$ $x - 1 = \frac{6}{2+e^t}$ $x = 1 + \frac{6}{2+e^t}$ **<u>Method 2 (Bring over)</u>** $x - 4 = (x-1)\left(-\frac{1}{2}\right)e^t$ $2x - 8 = (1-x)e^t$ $2x + xe^t = 8 + e^t$ $x = \frac{8+e^t}{2+e^t}$ $= 1 + \frac{6}{2+e^t}$	
(iii)	$\frac{3}{2} = 1 + \frac{6}{2+e^t}$ $\frac{1}{2+e^t} = \frac{1}{12}$ $2 + e^t = 12$ $t = \ln 10$	

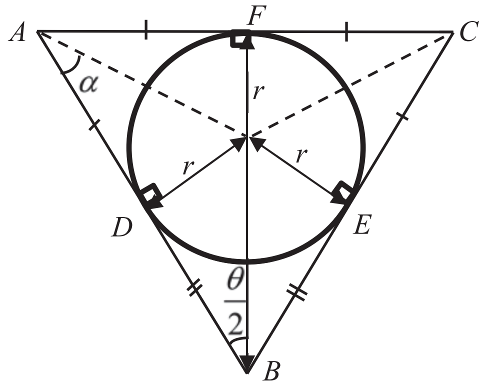
(iv)		

Qn	Solution	Mark Scheme
9	**Applications of Differentiation**	**[10]**
(i)	$\frac{dx}{dt} = v\cos\theta$ $\frac{dy}{dt} = v\sin\theta - 10t$ $\Rightarrow \frac{dy}{dx} = \frac{v\sin\theta - 10t}{v\cos\theta}$ For maximum height, $\frac{dy}{dx} = \frac{v\sin\theta - 10t}{v\cos\theta} = 0$ $\Rightarrow t = \frac{v\sin\theta}{10}.$ Hence, maximum height $= (v\sin\theta)\left(\frac{v\sin\theta}{10}\right) - 5\left(\frac{v\sin\theta}{10}\right)^2$ $= \frac{v^2\sin^2\theta}{20}$ metres. <u>Alternatively,</u> For maximum height, we have zero vertical velocity, i.e. $\frac{dy}{dt} = v\sin\theta - 10t = 0$ $\Rightarrow t = \frac{v\sin\theta}{10}.$ Hence, maximum height $= (v\sin\theta)\left(\frac{v\sin\theta}{10}\right) - 5\left(\frac{v\sin\theta}{10}\right)^2$ $= \frac{v^2\sin^2\theta}{20}$ metres.	
(ii)	Maximum height of particle $= 57.8 = \frac{v^2\sin^2\theta}{20} + 29$ $\Rightarrow v\sin\theta = 24$ since $v > 0$ and $0 < \theta < \frac{\pi}{2}.$ When particle hits ground, $y = -29 \Rightarrow -29 = (v\sin\theta)t - 5t^2$ $\Rightarrow 5t^2 - 24t - 29 = 0$ $\Rightarrow (5t - 29)(t + 1) = 0$ $\Rightarrow t = 5.8$ or $t = -1$ (reject, $t > 0$). Hence, time taken for particle to hit the ground is 5.8 seconds. When particle hits ground, $x = 104.4 \Rightarrow 104.4 = (v\cos\theta)t$ $\Rightarrow v\cos\theta = \frac{104.4}{5.8} = 18.$ Hence, $v^2\sin^2\theta + v^2\cos^2\theta = v^2$ $\Rightarrow 24^2 + 18^2 = v^2 = 900$ $\Rightarrow v = 30$ since $v > 0.$	

(iii)	When particle hits ground, $v\sin\theta = 24,\ v\cos\theta = 18,\ t = 5.8.$ Hence, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\sin\theta - 10t}{v\cos\theta} = \dfrac{24-10(5.8)}{18} = -\dfrac{17}{9}.$	

Qn	Solution	Mark Scheme
10	**APGP**	**[13]**
(i)	$a = 350$ $T_{20} = 65 \Rightarrow 350r^{19} = 65$ $\Rightarrow r = \left(\frac{65}{350}\right)^{\frac{1}{19}}$ $S_\infty = \frac{350}{1-\left(\frac{65}{350}\right)^{\frac{1}{19}}} = 4127.58 < 4128$ cm Since the sum of the infinite series is 4127.58 cm, hence the total length of all the planks will always be less than 4128 cm, no matter how many planks there are.	
(ii)	$L = \frac{350\left(1-\left(\frac{65}{350}\right)^{\frac{20}{19}}\right)}{1-\left(\frac{65}{350}\right)^{\frac{1}{19}}}$ $= 3426.029708$ $= 3426$ (nearest cm)	
(iii)	$S_{20} = L$ $3426.029708 = \frac{20}{2}(2a+19d)$ ..... (1) $T_{20} = 65 = a+19d$ ..... (2) $(1)-10(2)$: $2776.029708 = 10a$ $a = 277.6029708$ $= 278$ cm (3 s.f) (2): $d = -11.18963004$ $= -11.2$ cm (3 s.f)	
	$a = 65,\ d = 11$ Let S_N be the total length of the first N planks of the roof of house C $S_N > 20640$ $\frac{N}{2}[2(65)+(N-1)11] > 20640$ Using GC, When $N = 56,\ S_{56} = 20580 < 20640$ When $N = 57,\ S_{57} = 21261 > 20640$ Least value of $N = 57$	

	Alternative: $S_N > 20640$ $\frac{N}{2}[2(65)+(N-1)11] > 20640$ $130N + 11N^2 - 11N > 41280$ $11N^2 + 119N - 41280 > 0$ Using G.C., $N > 56.09$ Least value of $N = 57$	

Qn	Solution	Mark Scheme
11	**Applications of Differentiation**	[15]
(i)	 $\tan\frac{\theta}{2}=\frac{r}{DB}$ $DB=\frac{r}{\tan\frac{\theta}{2}}$ $=r\cot\frac{\theta}{2}$ (shown)	
(ii)	$\alpha=\frac{\pi-\theta}{4}=\frac{\pi}{4}-\frac{\theta}{4}$ $\tan\alpha=\frac{r}{AD}\Rightarrow AD=\frac{r}{\tan\alpha}$ $AD=\frac{r}{\tan\left(\frac{\pi}{4}-\frac{\theta}{4}\right)}$ $=r\cot\left(\frac{\pi}{4}-\frac{\theta}{4}\right)$ Let P be the perimeter of the triangle. $P=4AD+2BD$ $=4r\cot\left(\frac{\pi}{4}-\frac{\theta}{4}\right)+2r\cot\frac{\theta}{2}$ (shown)	
(iii)	$\frac{dP}{d\theta}=-4r\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\left(-\frac{1}{4}\right)-2r\operatorname{cosec}^2\frac{\theta}{2}\left(\frac{1}{2}\right)$ $=r\left(\frac{1}{\sin^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)}-\frac{1}{\sin^2\frac{\theta}{2}}\right)$ When $\frac{dP}{d\theta}=0$, $r\left(\frac{1}{\sin^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)}-\frac{1}{\sin^2\frac{\theta}{2}}\right)=0$ $\sin^2\frac{\theta}{2}-\sin^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)=0$	

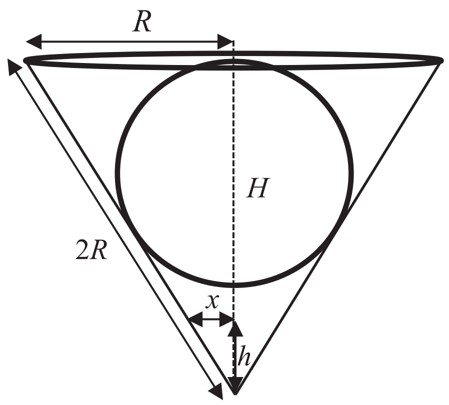
	$\sin^2\frac{\theta}{2}-\frac{1}{2}\left(1-\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)=0$ $\sin^2\frac{\theta}{2}+\frac{1}{2}\sin\frac{\theta}{2}-\frac{1}{2}=0$ $\sin\frac{\theta}{2}=\frac{1}{2}$ or $\sin\frac{\theta}{2}=-1$ $\frac{\theta}{2}=\frac{\pi}{6}$ $\frac{\theta}{2}=\frac{3\pi}{2}$ (Rejected $\because 0<\frac{\theta}{2}<\pi$) $\theta=\frac{\pi}{3}$	

θ	$\left(\frac{\pi}{3}\right)^-$	$\frac{\pi}{3}$	$\left(\frac{\pi}{3}\right)^+$
$\frac{\mathrm{d}P}{\mathrm{d}\theta}$	<0	0	>0

Hence, $\theta=\frac{\pi}{3}$ gives the minimum perimeter of the triangle.

$$\text{Minimum } P=4r\cot\left(\frac{\pi}{4}-\frac{\theta}{4}\right)+2r\cot\frac{\theta}{2}$$
$$=4r\cot\left(\frac{\pi}{6}\right)+2r\cot\left(\frac{\pi}{6}\right)$$
$$=6\sqrt{3}r \text{ cm}$$
$\therefore a=6,\ b=3$

(iv)



By Pythagoras Theorem, $H=\sqrt{(2R)^2-R^2}=\sqrt{3}R\ (\because R>0)$

By Similar Triangles, $\frac{x}{h}=\frac{R}{\sqrt{3}R}=\frac{1}{\sqrt{3}}\Rightarrow x=\frac{h}{\sqrt{3}}$

Let the volume of liquid in the inverted cone be V cm^3.

$$V=\frac{1}{3}\pi x^2 h$$
$$=\frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$$
$$=\frac{\pi}{9}h^3$$

	$\frac{dV}{dh}=\frac{\pi}{3}h^2$ When $V=24\pi$, $\frac{\pi}{9}h^3=24\pi \Rightarrow h=6$ By Chain Rule, $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$ $2=\frac{\pi}{3}(6)^2\times\frac{dh}{dt}$ $\frac{dh}{dt}=\frac{1}{6\pi}$ cm/min	

2019 H2 MATH (9758/02) JC 2 PRELIMINARY EXAMINATION – SOLUTIONS

Qn	Solution	Mark Scheme
1	**Maclaurin Series**	[5]
	$y=\sqrt{\left(5-\mathrm{e}^{2x}\right)}$ $y^2=5-\mathrm{e}^{2x}$ Differentiate w.r.t. x, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=-2\mathrm{e}^{2x}$ $y\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^{2x}$ (Shown) Differentiate w.r.t. x, $y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-2\mathrm{e}^{2x}$ When $x=0$, $y=2,\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{2},\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\dfrac{9}{8}$ $y=2-\dfrac{1}{2}x-\dfrac{9x^2}{8(2!)}+\ldots$ $\therefore y=2-\dfrac{1}{2}x-\dfrac{9x^2}{16}+\ldots$	

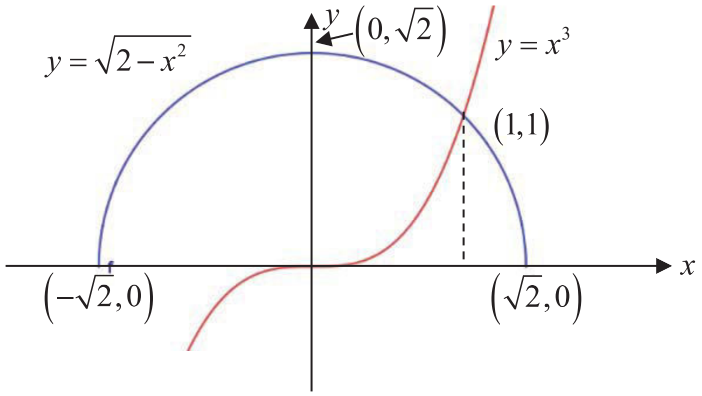
Qn	Solution	Mark Scheme
2	**Complex Numbers**	[9]
(i)	$z^2 - \mathrm{i}kz - 1 = 0$ $z = \dfrac{\mathrm{i}k \pm \sqrt{(-\mathrm{i}k)^2 - 4(1)(-1)}}{2(1)}$ $= \dfrac{\mathrm{i}k \pm \sqrt{4-k^2}}{2}$ Since $0 < \arg(z_1) < \arg(z_2) < \pi$, $z_1 = \dfrac{\sqrt{4-k^2}}{2} + \mathrm{i}\dfrac{k}{2}$ $z_2 = -\dfrac{\sqrt{4-k^2}}{2} + \mathrm{i}\dfrac{k}{2}$ <u>Alternatively</u> $z^2 - \mathrm{i}kz - 1 = 0$ $\left(z - \dfrac{\mathrm{i}k}{2}\right)^2 - \left(-\dfrac{\mathrm{i}k}{2}\right)^2 - 1 = 0$ $\left(z - \dfrac{\mathrm{i}k}{2}\right)^2 - \left(1 - \dfrac{k^2}{4}\right) = 0$ $\Rightarrow z = \dfrac{\mathrm{i}k}{2} \pm \sqrt{1 - \dfrac{k^2}{4}}$ $\Rightarrow z_1 = \sqrt{1-\dfrac{k^2}{4}} + \dfrac{\mathrm{i}k}{2}$ or $z_2 = -\sqrt{1-\dfrac{k^2}{4}} + \dfrac{\mathrm{i}k}{2}$ since $0 < \arg(z_1) < \arg(z_2) < \pi$. <u>Alternatively</u> Let $z = x + \mathrm{i}y$. $z^2 - \mathrm{i}kz - 1 = 0$ $(x+\mathrm{i}y)^2 - \mathrm{i}k(x+\mathrm{i}y) - 1 = 0$ $\Rightarrow (x^2 - y^2 + ky - 1) + \mathrm{i}(2xy - kx) = 0$ Equating real and imaginary parts, $x^2 - y^2 + ky - 1 = 0$ and $2xy - kx = 0$. $2xy - kx = 0 \Rightarrow x = 0$ or $y = \dfrac{k}{2}$. Reject $x = 0$, since $0 < \arg(z_1) < \arg(z_2) < \pi$ and hence z_1 and z_2 cannot be both purely imaginary ($\arg(z_1) \neq \arg(z_2)$). [OR: If $x = 0$, $-y^2 + ky - 1 = 0$ but discriminant $= k^2 - 4(-1)(-1) < 0$ since $0 < k < 2$ (which implies that y is not real). Hence, $x \neq 0$.]	

	$y=\frac{k}{2} \Rightarrow x^2-\left(\frac{k}{2}\right)^2+k\left(\frac{k}{2}\right)-1=0$ $\Rightarrow x=\pm\sqrt{1-\frac{k^2}{4}}$ $\Rightarrow z_1=\sqrt{1-\frac{k^2}{4}}+\frac{\mathrm{i}k}{2}$ or $z_2=-\sqrt{1-\frac{k^2}{4}}+\frac{\mathrm{i}k}{2}$ since $0<\arg(z_1)<\arg(z_2)<\pi$.	
(ii)	$\arg\left(\frac{z_1}{w}\right)=\frac{7\pi}{6}$ $\theta-\arg(w)=\frac{7\pi}{6}$ ------- (1) Since wz_1 purely imaginary, $\arg(w)+\theta=\frac{\pi}{2}$ or $\arg(w)+\theta=-\frac{\pi}{2}$ Substituting into (1) and solving, $\theta=\frac{5\pi}{6}$ or $\theta=\frac{\pi}{3}$ $\left(\text{rej since } 0<\theta<\frac{\pi}{2}\right)$ (Shown) <u>Alternatively</u> Since wz_1 purely imaginary, $\arg(w)=(2n+1)\frac{\pi}{2}$ where $n\in\mathbb{Z}$. Substituting into (1) and solving, $\theta=\pi\left(\frac{n}{2}+\frac{5}{6}\right)$. Since $0<\theta<\frac{\pi}{2}$, using GC, $n=-1$ so that $\theta=\frac{\pi}{3}$. (shown) <u>Alternatively</u> Since $0<\theta<\frac{\pi}{2}$, and $\theta-\arg(w)=\frac{7\pi}{6}$, we have $\arg(w)<-\frac{2\pi}{3}$. Hence, $\theta+\arg(w)<-\frac{\pi}{6}<0\Rightarrow\theta+\arg(w)=-\frac{\pi}{2}$. Solving, $\theta=\frac{\pi}{3}$. (shown)	

(iii)	$\tan^{-1}\left(\dfrac{\frac{k}{2}}{\frac{\sqrt{4-k^2}}{2}}\right)=\dfrac{\pi}{3}$ $\dfrac{k}{\sqrt{4-k^2}}=\sqrt{3}$ $k^2=12-3k^2$ $4k^2=12$ $k=\sqrt{3} \qquad (\text{since } k>0)$ $z_2=-\dfrac{\sqrt{4-\left(\sqrt{3}\right)^2}}{2}+\mathrm{i}\dfrac{\left(\sqrt{3}\right)}{2}$ $=-\dfrac{1}{2}+\mathrm{i}\dfrac{\sqrt{3}}{2}$ <u>Alternatively</u> Since $\arg(z_1)=\dfrac{\pi}{3}$, we have $\arg(z_2)=\pi-\dfrac{\pi}{3}=\dfrac{2\pi}{3}$. $\lvert z_2\rvert=\sqrt{\left(-\dfrac{\sqrt{4-k^2}}{2}\right)^2+\left(\dfrac{k}{2}\right)^2}=1$ $\Rightarrow z_2=1\left(\cos\left(\dfrac{2\pi}{3}\right)+\mathrm{i}\sin\left(\dfrac{2\pi}{3}\right)\right)=-\dfrac{1}{2}+\mathrm{i}\dfrac{\sqrt{3}}{2}$.	

Qn	Solution	Mark Scheme
3	**Vectors (Lines and Planes)**	**[13]**
(i)	$\begin{pmatrix}1\\2\\8\end{pmatrix}-\begin{pmatrix}1\\-4\\2\end{pmatrix}=\begin{pmatrix}0\\6\\6\end{pmatrix}=6\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\therefore$ vector perpendicular to $p_1=\begin{pmatrix}0\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}-3\\1\\-1\end{pmatrix}$ Equation of p_1 $\Rightarrow \mathbf{r}\bullet\begin{pmatrix}-3\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\2\\8\end{pmatrix}\bullet\begin{pmatrix}-3\\1\\-1\end{pmatrix}=-9$ $\Rightarrow \mathbf{r}\bullet\begin{pmatrix}3\\-1\\1\end{pmatrix}=9$ $\Rightarrow 3x-y+z=9$ (shown)	
(ii)	Since F is on line $l \Rightarrow \overrightarrow{OF}=\begin{pmatrix}1+\lambda\\-4+\lambda\\2-2\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. $\overrightarrow{AF}=\begin{pmatrix}1+\lambda\\-4+\lambda\\2-2\lambda\end{pmatrix}-\begin{pmatrix}1\\2\\8\end{pmatrix}=\begin{pmatrix}\lambda\\-6+\lambda\\-6-2\lambda\end{pmatrix}$ $\overrightarrow{AF}$ perpendicular to line $l \Rightarrow \begin{pmatrix}\lambda\\-6+\lambda\\-6-2\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\1\\-2\end{pmatrix}=0$ $\lambda-6+\lambda+12+4\lambda=0$ $6\lambda+6=0$ $\lambda=-1$ $\therefore \overrightarrow{OF}=\begin{pmatrix}0\\-5\\4\end{pmatrix}$ Coordinates form of $F=(0,-5,\ 4)$	

(iii)	$\overrightarrow{AF}=\begin{pmatrix}-1\\-7\\-4\end{pmatrix}$ Area of triangle $ABF=\dfrac{1}{2}\left\|\begin{pmatrix}-1\\-7\\-4\end{pmatrix}\times\begin{pmatrix}0\\6\\6\end{pmatrix}\right\|$ $=\dfrac{\|-6\|}{2}\left\|\begin{pmatrix}1\\7\\4\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}\right\|$ $=3\left\|\begin{pmatrix}3\\-1\\1\end{pmatrix}\right\|$ $=3\sqrt{11}$ units^2	
(iv)	Observe $\begin{pmatrix}0\\0\\9\end{pmatrix}$ is in p_1 $\Rightarrow\begin{pmatrix}0\\0\\9\end{pmatrix}-\begin{pmatrix}-1\\6\\6\end{pmatrix}=\begin{pmatrix}1\\-6\\3\end{pmatrix}$ $\therefore$ shortest distance C to p_1 $=\dfrac{\left\|\begin{pmatrix}1\\-6\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix}\right\|}{\sqrt{11}}=\dfrac{12}{\sqrt{11}}$ Volume of tetrahedron $ABFC=\dfrac{1}{3}\times\dfrac{12}{\sqrt{11}}\times3\sqrt{11}=12$ units^3	
(v)	$\overrightarrow{OD}=\dfrac{\overrightarrow{OC}+3\overrightarrow{OA}}{4}$ $=\dfrac{1}{4}\left[\begin{pmatrix}-1\\6\\6\end{pmatrix}+3\begin{pmatrix}1\\2\\8\end{pmatrix}\right]$ $=\dfrac{1}{4}\begin{pmatrix}2\\12\\30\end{pmatrix}$ $=\dfrac{1}{2}\begin{pmatrix}1\\6\\15\end{pmatrix}$ Equation of p_2 $\Rightarrow\mathbf{r}\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix}=\dfrac{1}{2}\begin{pmatrix}1\\6\\15\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix}$ $\Rightarrow\mathbf{r}\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix}=6$ $\Rightarrow 3x-y+z=6$	

Qn	Solution	Mark Scheme
4	**Applications of Integration**	[13]
		
(i)	Area $=\int_0^1 x^3\ \mathrm{d}x+\int_1^{\sqrt{2}}\sqrt{2-x^2}\ \mathrm{d}x$ $=\left[\frac{x^4}{4}\right]_0^1+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{2-\left(\sqrt{2}\sin\theta\right)^2}\ \left(\sqrt{2}\cos\theta\right)\ \mathrm{d}\theta$ $=\frac{1}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{2\left(1-\sin^2\theta\right)}\ \left(\sqrt{2}\cos\theta\right)\ \mathrm{d}\theta$ $=\frac{1}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{2\cos^2\theta}\ \left(\sqrt{2}\cos\theta\right)\ \mathrm{d}\theta$ $=\frac{1}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}2\cos^2\theta\ \mathrm{d}\theta$ $=\frac{1}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 2\theta+1\ \mathrm{d}\theta$ $=\frac{1}{4}+\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $=\frac{1}{4}+\left[\left(\frac{1}{2}\sin 2\left(\frac{\pi}{2}\right)+\frac{\pi}{2}\right)-\left(\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\right)\right]$ $=\frac{1}{4}+\left(\frac{\pi}{2}-\frac{1}{2}-\frac{\pi}{4}\right)$ $=\frac{\pi}{4}-\frac{1}{4}$	

(ii)

When $y = x^3$, $\dfrac{dy}{dx} = 3x^2$

$$S_1 = \int_0^1 2\pi\left(x^3\right)\sqrt{1+\left(3x^2\right)^2}\ dx$$

$$= \int_0^1 2\pi\left(x^3\right)\sqrt{1+9x^4}\ dx$$

$$= \frac{\pi}{18}\int_0^1 36x^3\sqrt{1+9x^4}\ dx$$

$$= \frac{\pi}{18}\left[\frac{\left(1+9x^4\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \frac{\pi}{27}\left(10^{\frac{3}{2}}-1\right)$$

When $y = \sqrt{2-x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{2-x^2}}$

$$S_2 = \int_1^{\sqrt{2}} 2\pi\sqrt{2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{2-x^2}}\right)^2}\ dx$$

$$= \int_1^{\sqrt{2}} 2\pi\sqrt{2-x^2}\sqrt{1+\frac{x^2}{2-x^2}}\ dx$$

$$= \int_1^{\sqrt{2}} 2\pi\sqrt{2-x^2}\ \frac{\sqrt{2}}{\sqrt{2-x^2}}\ dx$$

$$= 2\sqrt{2}\pi\int_1^{\sqrt{2}} 1\ dx$$

$$= 2\sqrt{2}\pi\left[x\right]_1^{\sqrt{2}}$$

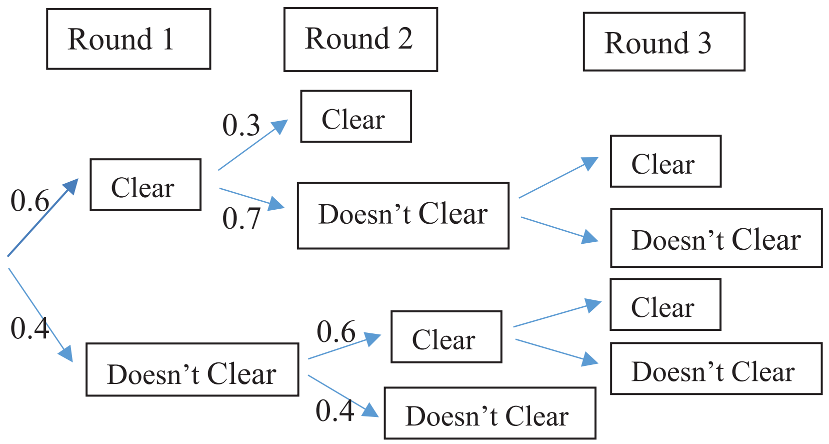
$$= 2\sqrt{2}\pi\left[\sqrt{2}-1\right]$$

$$= 4\pi - 2\sqrt{2}\pi$$

Total surface area $= \dfrac{\pi}{27}\left(10^{\frac{3}{2}}-1\right) + 4\pi - 2\sqrt{2}\pi$

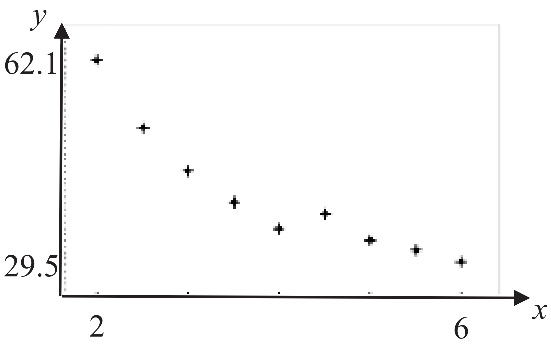
$$= \frac{\pi}{27}\left(10^{\frac{3}{2}}\right) + \frac{107\pi}{27} - 2\sqrt{2}\pi$$

Qn	Solution	Mark Scheme
5	**Discrete Random Variables**	**[6]**
(i)	$u+v+u+v=1 \Rightarrow v=0.5-u$ $\mathrm{E}(X)=2u+3v+4u+5v$ $=6u+8v$ $=6u+8(0.5-u)$ $=4-2u$	
(ii)	$\mathrm{E}(X^2)=2^2u+3^2v+4^2u+5^2v$ $=20u+34v$ $=20u+34(0.5-u)$ $=20u+17-34u$ $=17-14u$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=(17-14u)-(4-2u)^2$ $=1.16$ Using GC, $u=0.1$ or $u=0.4$. Since $u>v$, $u=0.4$ and $v=0.1$.	

Qn	Solution	Mark Scheme
6	**P&C and Probability**	**[9]**
(i)	 $\mathrm{P}(\text{a player plays 3 rounds})$ $=\mathrm{P}(\text{clears round 1 but does not clear round 2})$ $+\mathrm{P}(\text{does not clear round 1 but clears round 2})$ $=(0.6)(0.7)+(0.4)(0.6)=0.66$	
(ii)	$\mathrm{P}(\text{clears round 1} \mid \text{does not clear round 2})$ $=\dfrac{\mathrm{P}(\text{clears round 1 and does not clear round 2})}{\mathrm{P}(\text{does not clear round 2})}$ $=\dfrac{(0.6)(0.7)}{(0.6)(0.7)+(0.4)(0.4)}$ $=\dfrac{21}{29}=0.724\ (3\text{ s.f})$	
(iii)	$\mathrm{P}(\text{a player clears exactly 2 rounds})$ $=\mathrm{P}(\text{clears round 1 and round 2})$ $+\mathrm{P}(\text{clears round 1, does not clear round 2, clears round 3})$ $+\mathrm{P}(\text{does not clear round 1, clears round 2 and round 3})$ $=(0.6)(0.3)+0.2$ $=0.38$	
(iv)	Number of ways for last digit = 5 Number of ways required = $9 \times 8 \times 7 \times 6 \times 5 \times 5 = 75600$	
(v)	Number of ways for 3 odd digits = ${}^5C_3 = 10$ Number of ways for 3 even digits = ${}^5C_3 = 10$ Number of ways required = $10 \times 10 \times 6! = 72000$	

Qn	Solution	Mark Scheme
7	**Hypothesis Testing**	**[10]**
	Let X be the weight of a randomly chosen mini bread (in grams). Let μ denote the population mean weight of mini breads (in grams) Unbiased estimate of population mean, $\bar{x} = \frac{3571}{80}$ $= 44.6375$ Unbiased estimate of population variance, $s^2 = \frac{1}{79}\left(159701 - \frac{(3571)^2}{80}\right)$ $= 3.8036$ $= 3.80$ (3 s.f.)	
	$H_0: \mu = 45$ $H_1: \mu < 45$ Under H_0, since $n = 80$ is large, by Central Limit Theorem, $\overline{X} \sim N\left(45, \frac{3.8036}{80}\right)$ approximately Test Statistic: $Z = \dfrac{\overline{X} - 45}{\sqrt{\frac{3.8036}{80}}}$ Level of significance : 4 % Reject H_0 if p - value < 0.04 Using GC, p-value = 0.0482 Since p-value = $0.0482 > 0.04$, we do not reject H_0 and conclude that there is insufficient evidence, at the 4% level of significance, that the population mean weight is less than 45 grams. Thus, the customer's claim is not supported at the 4% significance level.	
	Sample mean based on combined sample $= \frac{\sum x + 20k}{80 + 20} = \frac{3571 + 20k}{100}$ $H_0: \mu = 45$ $H_1: \mu < 45$ Under H_0, $X \sim N(45, 1.5^2) \Rightarrow \overline{X} \sim N\left(45, \frac{1.5^2}{100}\right)$ Test statistic: $Z = \dfrac{\overline{X} - 45}{\sqrt{\frac{1.5^2}{100}}}$ Level of significance : 4 % Reject H_0 if z − value < -1.7507 $z - \text{value} = \dfrac{\bar{x} - 45}{\sqrt{\frac{1.5^2}{100}}}$	

	Since there is sufficient evidence that the customer's claim is valid at 4% level of significance, H_0 is rejected $\dfrac{\overline{x}-45}{\sqrt{\dfrac{1.5^2}{100}}}<-1.7507$ $\overline{x}<44.737395$ $\dfrac{3571+20k}{100}<44.737395$ $k<45.137$ $k<45.1 \quad (3\text{ s.f})$	

Qn	Solution	Mark Scheme
8	**Correlation and Regression**	**[10]**
(i)	$\bar{x} = \dfrac{2.0+2.5+3.0+3.5+4.0+4.5+5.0+5.5+6.0}{9} = 4$ $\bar{y} = 68.8067 - 7.12667(4) = 40.30002$ $\dfrac{62.1+51.2+44.1+39.1+35+k+33+31.4+29.5}{9} = 40.30002$ $k = 37.30018$ $= 37.3$ (to 1 d.p) (shown)	
(ii)		
(iii)	Model **(A)**: $r = -0.922$ Model **(B)**: $r = -0.866$ Model **(C)**: $r = 0.990$ Since the value of $\lvert r \rvert = 0.990$ for Model (C) is closest to 1, Model (C) is the best model. Using GC, equation of suitable regression line: $y = 13.685 + \dfrac{94.313}{x}$ $y = 13.7 + \dfrac{94.3}{x}$ (3 s.f)	
(iv)	When $x = 4.2$, $y = 13.685 + \dfrac{94.313}{4.2} = 36.1$ (3 s.f) Since $x = 4.2$ is within the data range of x **and** $\lvert r \rvert = 0.990$ is close to 1, the estimated reaction time is reliable.	

Qn	Solution	Mark Scheme
9	**Normal and Sampling Distributions**	**[12]**
(i)	Let A be the mass of a randomly chosen Alpha apple in grams. Given $A \sim \mathrm{N}\left(\mu, \sigma^2\right)$ $\overline{A} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{8}\right)$ and $A_1 + A_2 + A_3 + \ldots + A_8 \sim \mathrm{N}\left(8\mu, 8\sigma^2\right)$ $\mathrm{P}\left(A_1 + A_2 + \ldots + A_8 > 3000\right) = 0.5 \Rightarrow 8\mu = 3000 \Rightarrow \mu = 375$ $\mathrm{P}\left(\overline{A} < 370\right) = 0.25$ $\mathrm{P}\left(Z < \dfrac{370-\mu}{\sqrt{\dfrac{\sigma^2}{8}}}\right) = 0.25$ $\dfrac{370-375}{\sqrt{\dfrac{\sigma^2}{8}}} = -0.67449$ $\sigma = 21.0$ (3s.f)	
(ii)	$A \sim \mathrm{N}\left(380, 20^2\right)$ Let $S = A_1 + A_2 + \ldots + A_8 \sim \mathrm{N}\left(8\times 380, 8\times 20^2\right) = \mathrm{N}\left(3040, 3200\right)$ $\mathrm{P}\left(2900 < S < 3100\right) = 0.849$ (3s.f.)	
(iii)	Let B be the mass of a randomly chosen Beta apple in grams. $B \sim \mathrm{N}\left(250, 18^2\right)$ Let $T = B_1 + B_2 + \ldots + B_{12} \sim \mathrm{N}\left(12\times 250, 12\times 18^2\right) = \mathrm{N}\left(3000, 3888\right)$. Let $C = 0.7S + 0.8T$. $\mathrm{E}(C) = 0.7\times 3040 + 0.8\times 3000 = 4528$ $\mathrm{Var}(C) = 0.7^2 \times 3200 + 0.8^2 \times 3888 = 4056.32$ $\therefore C \sim \mathrm{N}\left(4528, 4056.32\right)$ $\mathrm{P}\left(C > 4500\right) = 0.670$ (3 s.f)	
(iv)	Assume that the distributions of the masses of all apples are independent of one another.	
	$p = \mathrm{P}\left(0.9C > 4100\right) = \mathrm{P}\left(C > 4555.6\right) < \mathrm{P}\left(C > 4500\right)$ Thus p is lower than the answer in part **(iii)**.	

Qn	Solution	Mark Scheme
10	**Binomial Distribution and Probability**	**[13]**
(a)	The **conditional** probability, p, that a randomly chosen sweet is red is not the same for the 1st to the 5th sweets. For example, for the first sweet, $p=\frac{6}{10}$. For the second sweet, $p=\frac{5}{9}$ if the first sweet is red and $p=\frac{6}{9}$ if the first sweet is not red. Hence, whether a randomly chosen sweet is red or not is **not independent** of other sweets.	
(b)(i)	Let X be the no of sweets, out of 10, that are red. $X \sim \mathrm{B}(10,p)$ Given that $\mathrm{P}(X=5)=0.21253$, $\binom{10}{5} p^5(1-p)^5=0.21253$ $p^5(1-p)^5=0.00084337$ $p(1-p)=0.24277$ $k=0.24277=0.243 \quad (3\text{ s.f.})$ Using GC, $p=0.415 \quad \text{or} \quad 0.585 \quad (3\text{ s.f.})$	
(ii)	$X \sim \mathrm{B}(10,0.6)$ $\mathrm{P}(X \le 8 \mid X>2)$ $=\frac{\mathrm{P}(2<X\le 8)}{\mathrm{P}(X>2)}$ $=\frac{\mathrm{P}(X\le 8)-\mathrm{P}(X\le 2)}{1-\mathrm{P}(X\le 2)}$ $=\frac{0.94135}{0.98771}$ $=0.953$	
(iii)	Required Probability $=\mathrm{P}(X\le 5)\times\mathrm{P}(X\ge 5)\times 2-\mathrm{P}(X=5)^2$ $=\mathrm{P}(X\le 5)\times(1-\mathrm{P}(X\le 4))\times 2-\mathrm{P}(X=5)^2$ $=0.572$ **OR** Required Probability $=\mathrm{P}(X\le 5)\times\mathrm{P}(X>5)\times 2+\mathrm{P}(X=5)\times\mathrm{P}(X<5)\times 2+\mathrm{P}(X=5)^2$ $=0.572$	

(iv)	$\mathrm{P}\left(\lvert X_1 - X_2\rvert \geq 8\right)$ $= \left[\underbrace{\mathrm{P}\left(X_1 = 10\right)\times\mathrm{P}\left(X_2 \leq 2\right)}_{\text{Consider 3 cases}} + \underbrace{\mathrm{P}\left(X_1 = 9\right)\times\mathrm{P}\left(X_2 \leq 1\right)}_{\text{Consider 2 cases}} + \mathrm{P}\left(X_1 = 8\right)\times\mathrm{P}\left(X_2 = 0\right)\right]\times 2$ $= 0.000309$	