2022 C1 Block Test Revision Package Solutions
Chapter 6A Techniques of Integration

1(a)	$\int(\cos 2x\sin 7x)\,dx = \int(\sin 7x\cos 2x)\,dx$ $= \frac{1}{2}\int(\sin 9x+\sin 5x)\,dx$ $= -\frac{1}{2}\left(\frac{\cos 9x}{9}+\frac{\cos 5x}{5}\right)+C$ This is a product of two trigo terms. Use MF26 factor formula to convert it into two trigo terms to integrate easily.
1(b)	<u>Method 1: Split the Numerator</u> $\int\frac{4x-6}{x^2+6x}\,dx = 2\int\frac{2x+6}{x^2+6x}\,dx-18\int\frac{1}{x^2+6x}\,dx$ $= 2\ln\left\|x^2+6x\right\|-18\int\frac{1}{(x+3)^2-3^2}\,dx$ $=2\ln\left\|x^2+6x\right\|-18\times\frac{1}{2(3)}\ln\left\|\frac{x}{x+6}\right\|+C$ $=2\ln\left\|x^2+6x\right\|-3\ln\left\|\frac{x}{x+6}\right\|+C$ Note that standard form $\frac{f\,'}{f}$ does not work here. Thus have to use split numerator method. <u>Method 2: By Partial Fractions</u> $\int\frac{4x-6}{x^2+6x}\,dx = \int\left(-\frac{1}{x}+\frac{5}{x+6}\right)dx$ $=-\ln\left\|x\right\|+5\ln\left\|x+6\right\|+C$ Since denominator of integral can be factorised, we can use partial fraction here
1(c)	$\int e^x\tan^{-1}(e^x)\,dx = e^x\tan^{-1}(e^x)-\int e^x\frac{e^x}{1+e^{2x}}\,dx$ $=e^x\tan^{-1}(e^x)-\int\frac{e^{2x}}{1+e^{2x}}\,dx$ $=e^x\tan^{-1}(e^x)-\left(\frac{1}{2}\right)\int\frac{2e^{2x}}{e^{2x}+1}\,dx$ $=e^x\tan^{-1}(e^x)-\frac{1}{2}\ln\left(e^{2x}+1\right)+C$
2(a)	$\int\sqrt{\text{cosec}2x-\sin 2x}\,dx = \int\sqrt{\frac{1}{\sin 2x}-\sin 2x}\,dx$ $=\int\sqrt{\frac{1-\sin^2 2x}{\sin 2x}}\,dx$

	$=\int \frac{\cos 2x}{\sqrt{\sin 2x}}\,dx$ $=\frac{2\sqrt{\sin 2x}}{2}+C$ $=\sqrt{\sin 2x}+C$	Try to find ways to have the terms under the square root to be squared. Note that $\int \frac{\cos 2x}{\sqrt{\sin 2x}}\,dx=\int \cos 2x(\sin 2x)^{-1/2}\,dx$ Can meet the standard form $f\,'(f)^n$ form.
2(b)	$\int \ln(4+x^2)\ dx = x\ln(4+x^2)-\int \frac{2x^2}{4+x^2}dx$ $=x\ln(4+x^2)-2\int 1-\frac{4}{4+x^2}dx$ $=x\ln(4+x^2)-2\left[x-\frac{4}{2}\tan^{-1}\frac{x}{2}\right]+C$ $=x\ln(4+x^2)-2x+4\tan^{-1}\frac{x}{2}+C$	Integrate ln terms along: think of By Parts method, with hidden partner as 1. $\int \frac{2x^2}{4+x^2}dx$ is not a proper fraction. Thus do long division before integrating.
3(a)	$\int x\left[(1-3x^2)^5+e^{x^2+1}\right]dx=\int x(1-3x^2)^5\,dx+\int xe^{x^2+1}\,dx$ $=-\frac{1}{6}\int(-6x)(1-3x^2)^5\,dx+\frac{1}{2}\int 2xe^{x^2+1}\,dx$ $=-\frac{(1-3x^2)^6}{36}+\frac{1}{2}e^{x^2+1}+C$	Note that $\int f'(x)e^{f(x)}\,dx=e^{f(x)}+C$
3(b)	$\frac{d}{dx}\cos^{-1}(x^2)=-\frac{2x}{\sqrt{1-x^4}}$ $\int_0^{\frac{1}{\sqrt{2}}}\frac{x}{\sqrt{1-x^4}}\,dx=-\frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{-2x}{\sqrt{1-x^4}}\,dx$ $=-\frac{1}{2}\left[\cos^{-1}(x^2)\right]_0^{\frac{1}{\sqrt{2}}}$ $=-\frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{2}\right)$ $=\frac{\pi}{12}$	Note that you are expected to give exact value. Know your trigo basic angles and their properties.
4(a) **(i)**	$\frac{d}{dx}\left(e^{x^2}\right)=2xe^{x^2}$	

4(a) (ii)	$\int_0^2 x^3 e^{x^2}\,dx = \int_0^2 \left(\frac{1}{2}x^2\right)\left(2xe^{x^2}\right)\,dx$ $= \left[\frac{1}{2}x^2 e^{x^2}\right]_0^2 - \int_0^2 xe^{x^2}\,dx$ $= 2e^4 - \left[\frac{1}{2}e^{x^2}\right]_0^2$ $= \frac{3}{2}e^4 + \frac{1}{2}$ Let $u = \frac{1}{2}x^2$ $\quad \frac{dv}{dx} = 2xe^{x^2}$ $\frac{du}{dx} = x$ $\quad v = e^{x^2}$ Do link this part to part (i). Thus take note of how By parts method is split for By Parts Method to be used.
4 (b)	$\int \frac{x+1}{x^2-6x+13}\,dx = \frac{1}{2}\int \frac{2x-6}{x^2-6x+13}\,dx + \int \frac{4}{x^2-6x+13}dx$ $= \frac{1}{2}\ln\left|x^2-6x+13\right| + 4\int \frac{1}{(x-3)^2+2^2}dx + C_1$ $= \frac{1}{2}\ln\left(x^2-6x+13\right) + 2\tan^{-1}\left(\frac{x-3}{2}\right) + C$ Use split numerator method. where C_1 , C are arbitrary constants and since $x^2-6x+13>0$.
4(c)(i)	$\int e^x \cos x\,dx = e^x\cos x - \int e^x(-\sin x)\,dx$ $= e^x \cos x + \int e^x \sin x\,dx$ $= e^x\cos x + \left[e^x \sin x - \int e^x \cos x\,dx\right]$ $2\int e^x \cos x\,dx = e^x\cos x + e^x \sin x$ $\int e^x\cos x\,dx = \frac{1}{2}\left(e^x\cos x + e^x \sin x\right) + c$ where c is an arbitrary constant. By parts method usually do max 2 rounds usually. Combine $\int e^x \cos x\,dx$ that appears on both sides. Let $u_1 = \cos x$ $\quad \frac{dv_1}{dx} = e^x$ $\frac{du_1}{dx} = -\sin x$ $\quad v_1 = e^x$ Let $u = \sin x$ $\quad \frac{dv}{dx} = e^x$ $\frac{du}{dx} = \cos x$ $\quad v = e^x$

4(c)(ii)

$$\int_0^{n\pi} \mathrm{e}^x \cos x \,\mathrm{d}x = \frac{1}{2}\left[\mathrm{e}^x \cos x + \mathrm{e}^x \sin x\right]_0^{n\pi}$$

$$= \frac{1}{2}\left[\mathrm{e}^{n\pi} \cos n\pi + \mathrm{e}^{n\pi} \sin n\pi\right] - \frac{1}{2}\left[\mathrm{e}^0 \cos(0) - \mathrm{e}^0 \sin(0)\right]$$

$$= \frac{1}{2}\left[\mathrm{e}^{n\pi} \cos n\pi + \mathrm{e}^{n\pi} \sin n\pi\right] - \frac{1}{2}(1)$$

When n is a positive odd integer, $\left[\mathrm{e}^{n\pi} \cos n\pi + \mathrm{e}^{n\pi} \sin n\pi\right] = \mathrm{e}^{n\pi}(-1) + \mathrm{e}^{n\pi}(0) = -\mathrm{e}^{n\pi}$

When n is a positive even integer , $\left[\mathrm{e}^{n\pi} \cos n\pi + \mathrm{e}^{n\pi} \sin n\pi\right] = \mathrm{e}^{n\pi}(1) + \mathrm{e}^{n\pi}(0) = \mathrm{e}^{n\pi}$

$$\int_0^{n\pi} \mathrm{e}^x \cos x \,\mathrm{d}x \begin{cases} = \frac{1}{2}\left(-\mathrm{e}^{n\pi} - 1\right) \text{ if } n \text{ is a postive odd integer or} \\ = \frac{1}{2}\left(\mathrm{e}^{n\pi} - 1\right) \text{ if } n \text{ is a positive even integer} \end{cases}$$

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$$\int x\mathrm{e}^x \mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x \mathrm{d}x + c_1$$

$$= x\mathrm{e}^x - \mathrm{e}^x + C$$

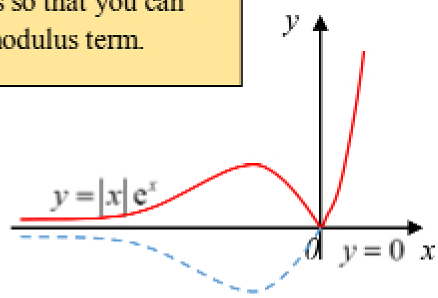
Split the limits so that you can simplify the modulus term.

$$\int_{-2}^{1} |x|\,\mathrm{e}^x \mathrm{d}x = \int_{-2}^{0} -x\mathrm{e}^x \mathrm{d}x + \int_0^1 x\mathrm{e}^x \mathrm{d}x$$

$$= \left[-x\mathrm{e}^x + \mathrm{e}^x\right]_{-2}^{0} + \left[x\mathrm{e}^x - \mathrm{e}^x\right]_0^1$$

$$= 1 - \left(2\mathrm{e}^{-2} + \mathrm{e}^{-2}\right) + (\mathrm{e} - \mathrm{e}) - (-1)$$

$$= 2 - 3\mathrm{e}^{-2}$$



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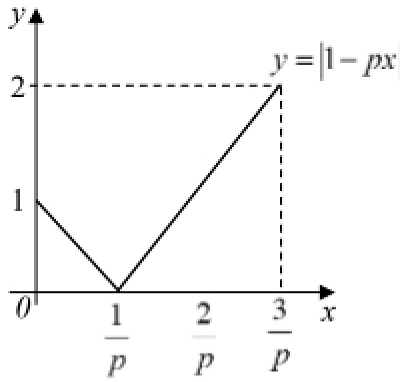
$$\int_{\frac{1}{p}}^{\frac{\sqrt{3}}{p}} \frac{1}{p^2x^2+1}\,\mathrm{d}x = \frac{1}{p^2}\int_{\frac{1}{p}}^{\frac{\sqrt{3}}{p}} \frac{1}{x^2 + \left(\frac{1}{p}\right)^2}\,\mathrm{d}x$$

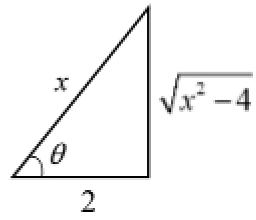
$$= \frac{1}{p^2}\frac{1}{\left(\frac{1}{p}\right)}\left[\tan^{-1}(px)\right]_{\frac{1}{p}}^{\frac{\sqrt{3}}{p}}$$

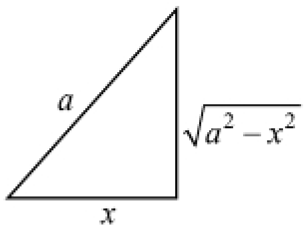
$$= \frac{1}{p}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\right]$$

$$= \frac{1}{p}\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{12p}$$

	$\int_0^{\frac{3}{p}} \lvert 1-px \rvert \,\mathrm{d}x = \int_0^{\frac{1}{p}} 1-px \,\mathrm{d}x + \int_{\frac{1}{p}}^{\frac{3}{p}} -(1-px)\,\mathrm{d}x$ $= \left[x-\frac{px^2}{2}\right]_0^{\frac{1}{p}} - \left[x-\frac{px^2}{2}\right]_{\frac{1}{p}}^{\frac{3}{p}}$ $= \frac{1}{p}-\frac{1}{2p}-\left(\frac{3}{p}-\frac{9}{2p}-\frac{1}{p}+\frac{1}{2p}\right)$ $= \frac{5}{2p}$  $\frac{\pi}{12p}=\frac{5k}{2p}$ $k=\frac{\pi}{30}$
7(a)	$\int \sin\left(\frac{3}{2}x\right)\cos\left(\frac{1}{2}x\right)\,\mathrm{d}x$ $=\frac{1}{2}\int \sin(2x)+\sin(x)\,\mathrm{d}x$ $=-\frac{\cos(2x)}{4}-\frac{\cos(x)}{2}+C$
7(b)	Since $y=\frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{x^2}$, when $x=\frac{4}{\sqrt{3}}$, $y=\frac{\sqrt{3}}{4}$ and $x=2$, $y=\frac{1}{2}$ $\int_2^{\frac{4}{\sqrt{3}}} \frac{1}{x\sqrt{x^2-4}}\,\mathrm{d}x$ $=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{4}} \frac{y}{\sqrt{\left(\left(\frac{1}{y}\right)^2-4\right)}}\cdot\left[-\left(\frac{1}{y}\right)^2\right]\mathrm{d}y$ Substitution method: Remember to change 3 areas: - x limits to y limits - The terms to integrate into y terms - Do differentiation to find equivalent y terms to replace dx.

	$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{4}} -\frac{1}{\sqrt{1-4y^2}}\,\mathrm{d}y$ $= \frac{1}{2}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{4}} -\frac{1}{\sqrt{\left(\left(\frac{1}{2}\right)^2 - y^2\right)}}\,\mathrm{d}y$ $= \frac{1}{2}\left[-\sin^{-1}(2y)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{4}}$ $= \frac{1}{2}\left(-\frac{\pi}{3} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{12}$
8(a)	$x = 2\sec\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec\theta\tan\theta$ $\int \frac{1}{x^3\sqrt{x^2-4}}\,\mathrm{d}x$ $= \int \frac{1}{8\sec^3\theta\sqrt{4\sec^2\theta - 4}}\, 2\sec\theta\tan\theta\,\mathrm{d}\theta$ $= \int \frac{1}{8\sec^3\theta(2\tan\theta)}\, 2\sec\theta\tan\theta\,\mathrm{d}\theta$ $= \int \frac{1}{8}\cos^2\theta\,\mathrm{d}\theta$ $= \frac{1}{16}\int(\cos 2\theta + 1)\,\mathrm{d}\theta$ $= \frac{1}{32}\sin 2\theta + \frac{1}{16}\theta + C$ $= \frac{1}{16}\sin\theta\cos\theta + \frac{1}{16}\theta + C$ $= \frac{\sqrt{x^2-4}}{8x^2} + \frac{1}{16}\cos^{-1}\left(\frac{2}{x}\right) + C$ - Remember to replace solution in θ terms back into x terms. - Make use of right angle triangle to find simplify trigo terms into x terms. 
8(bi)	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\tan^{-1}x\right)^2\right] = \frac{2\tan^{-1}x}{1+x^2}$
8(bii)	$\int \frac{x^3+x+1}{x^2+1}\,\mathrm{d}x = \int x + \frac{1}{x^2+1}\,\mathrm{d}x$ $= \frac{x^2}{2} + \tan^{-1}x + C$

8(biii)	Let $u=\tan^{-1}x$, $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{x^3+x+1}{x^2+1}$ Therefore, $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{1+x^2}$, $v=\frac{x^2}{2}+\tan^{-1}x$ - Ponder how you can make use of (b)(i) and (b)(ii) to solve this part. Integrating by parts, $\int\frac{\left(x^3+x+1\right)\tan^{-1}x}{x^2+1}\ \mathrm{d}x$ $=\tan^{-1}x\left[\frac{x^2}{2}+\tan^{-1}x\right]-\int\frac{1}{x^2+1}\left(\frac{x^2}{2}+\tan^{-1}x\right)\mathrm{d}x$ $=\frac{x^2}{2}\tan^{-1}x+\left(\tan^{-1}x\right)^2-\frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)\mathrm{d}x-\int\frac{\tan^{-1}x}{1+x^2}\ \mathrm{d}x$ $=\frac{x^2}{2}\tan^{-1}x+\left(\tan^{-1}x\right)^2-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x-\frac{1}{2}\left(\tan^{-1}x\right)^2+C$ $=\frac{x^2}{2}\tan^{-1}x+\frac{1}{2}\left(\tan^{-1}x\right)^2-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+C$
9(a)	$\int\frac{x^2}{\sqrt{a^2-x^2}}\mathrm{d}x$ $\qquad$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-a\sin\theta$ $=\int\frac{a^2\cos^2\theta}{a\sin\theta}\left(-a\sin\theta\right)\,\mathrm{d}\theta$ $=-a^2\int\cos^2\theta\ \mathrm{d}\theta$ $=-\frac{a^2}{2}\int\cos2\theta+1\ \mathrm{d}\theta$ $=-\frac{a^2}{2}[\frac{1}{2}\sin2\theta+\theta]+C$ $=-\frac{a^2}{2}[\cos\theta\sin\theta+\theta]+C$ $=-\frac{a^2}{2}\left(\left(\frac{x}{a}\times\frac{\sqrt{a^2-x^2}}{a}\right)+\cos^{-1}\left(\frac{x}{a}\right)\right)+C$ $=-\frac{x}{2}\left(\sqrt{a^2-x^2}\right)-\frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right)+C$ (Shown) - Know how to use right angle triangle to replace solution in θ terms back into x terms. 
9(bi)	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\cos(x)}=-\sin(x)\mathrm{e}^{\cos(x)}$

9(bii)	$$\begin{aligned}\int \sin(2x)e^{\cos x}\,dx &= \int 2\sin x\cos x\, e^{\cos x}dx \\ &= -2\int(\cos x)\left(-(\sin x)e^{\cos x}\right)dx \\ &= -2\left[(\cos x)e^{\cos x} - \int(-\sin x)e^{\cos x}dx\right] \\ &= -2\left[(\cos x)e^{\cos x} - e^{\cos x}\right] + C\end{aligned}$$	Notice the part that follows $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ and observe to select the form of the product of 2 terms for the By parts method.
10(i)	Method 1: $$\begin{aligned}u + \frac{1}{u} &= \frac{u^2+1}{u} \\ &= \frac{(\sec x + \tan x)^2 + 1}{\sec x + \tan x} \\ &= \frac{\sec^2 x + 2\sec x\tan x + (\tan^2 x + 1)}{\sec x + \tan x} \\ &= \frac{2\sec^2 x + 2\sec x\tan x}{\sec x + \tan x} \quad \text{since } 1 + \tan^2 x = \sec^2 x \\ &= \frac{2\sec x(\sec x + \tan x)}{\sec x + \tan x} \\ &= 2\sec x\end{aligned}$$ Method 2: $$\begin{aligned}u + \frac{1}{u} &= \sec x + \tan x + \frac{1}{\sec x + \tan x} \times \frac{\sec x - \tan x}{\sec x - \tan x} \\ &= \sec x + \tan x + \frac{\sec x - \tan x}{\sec^2 x - \tan^2 x} \\ &= \sec x + \tan x + \frac{\sec x - \tan x}{1} \quad \text{since } 1 + \tan^2 x = \sec^2 x \\ &= 2\sec x\end{aligned}$$	
10(ii)	$u = \sec x + \tan x$ $$\frac{du}{dx} = \sec x\tan x + \sec^2 x = (\sec x)(\sec x + \tan x)$$ When $x = 0$, $u = 1 + 0 = 1$. When $x = \frac{\pi}{6}$, $u = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$.	

Using result in (i) : $u+\dfrac{1}{u}=2\sec x$.

That is, $\sec x=\dfrac{1}{2}\left(u+\dfrac{1}{u}\right)$.

$$\int_0^{\frac{\pi}{6}}\frac{\sec^2 x}{\left(\sec x+\tan x\right)^3}\ \mathrm{d}x$$

$$=\int_1^{\sqrt{3}}\frac{\sec^2 x}{\left(\sec x+\tan x\right)^3}\ \frac{1}{(\sec x)(\sec x+\tan x)}\ \mathrm{d}u$$

$$=\frac{1}{2}\int_1^{\sqrt{3}}\frac{\left(u+\frac{1}{u}\right)}{u^4}\ \mathrm{d}u$$

$$=\frac{1}{2}\int_1^{\sqrt{3}}\left(\frac{1}{u^3}+\frac{1}{u^5}\right)\mathrm{d}u$$

$$=\frac{1}{2}\left[-\frac{1}{2u^2}-\frac{1}{4u^4}\right]_1^{\sqrt{3}}$$

$$=-\frac{1}{8}\left[\frac{2}{u^2}+\frac{1}{u^4}\right]_1^{\sqrt{3}}$$

$$=-\frac{1}{8}\left[\left(\frac{2}{3}+\frac{1}{9}\right)-3\right]$$

$$=\frac{5}{18}$$

This question can also be done directly without substitution:

$$\int_0^{\frac{\pi}{6}}\frac{\sec^2 x}{\left(\sec x+\tan x\right)^3}\ \mathrm{d}x$$

$$=\int_0^{\frac{\pi}{6}}\frac{\cos x}{\left(1+\sin x\right)^3}\ \mathrm{d}x$$

$$=\left[\frac{1}{(-2)\left(1+\sin x\right)^2}\right]_0^{\frac{\pi}{6}}$$

$$=\frac{5}{18}$$

11	$\int_0^{\frac{\pi}{6}} x\cos 2x\ \mathrm{d}x = \left[x\dfrac{\sin 2x}{2}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \dfrac{\sin 2x}{2}\ \mathrm{d}x$ $= \dfrac{\sqrt{3}}{4}\left(\dfrac{\pi}{6}\right) + \dfrac{1}{2}\left[\dfrac{\cos 2x}{2}\right]_0^{\frac{\pi}{6}}$ $= \dfrac{\sqrt{3}\pi}{24} + \dfrac{1}{2}\left(\dfrac{1}{4} - \dfrac{1}{2}\right)$ $= \dfrac{\sqrt{3}\pi}{24} - \dfrac{1}{8}$ $\int_0^{\frac{\pi}{6}} x\sin^2 x\ \mathrm{d}x = \int_0^{\frac{\pi}{6}} x\left(\dfrac{1-\cos 2x}{2}\right)\mathrm{d}x$ $= \dfrac{1}{2}\int_0^{\frac{\pi}{6}} \left(x - x\cos 2x\right)\mathrm{d}x$ $= \dfrac{1}{2}\left[\dfrac{x^2}{2}\right]_0^{\frac{\pi}{6}} - \dfrac{1}{2}\left(\dfrac{\sqrt{3}\pi}{24} - \dfrac{1}{8}\right)$ $= \dfrac{\pi^2}{144} - \dfrac{\sqrt{3}\pi}{48} + \dfrac{1}{16}$