**VICTORIA SCHOOL**
**S3 ELEMENTARY MATHEMATICS**

Name: ____Teacher's Solution_____

Date: ____________________

| Class | Reg No |
|---|---|
| **3** | |

**CHAPTER 1: QUADRATIC AND FRACTIONAL EQUATIONS**
**Reference Book: think! Mathematics Secondary Textbook, Shinglee**

In this chapter you will learn how to:

- Solve quadratic equations in one unknown variable by
  - Factorisation
  - Completing the square for $x^2 + px + q$
  - Use of formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the quadratic equation is of the form $ax^2 + bx + c = 0$ and $a$, $b$ and $c$ are real constants
  - Graphical Method (we will learn this later)
- Solve fractional equations that can be reduced to quadratic equations
- Formulate a quadratic equation in one variable to solve problems
- Sketch the graphs of quadratic equations of the form $y = (x-h)(x-k)$, $y = -(x-h)(x-k)$, $y = (x-p)^2 + q$ and $y = -(x-p)^2 + q$.

## 1 Introduction

A **quadratic expression** is in the form of $ax^2 + bx + c$, where $a, b, c$ are real numbers, and $a \neq 0$.

### 1.1 Solving Quadratic Equations by Factorisation (Recap)

Last year, we learnt how to solve a quadratic equation by using the **Zero Product Property:**

$$\text{If} \quad A \times B = 0, \quad \text{then either } A = 0 \text{ or } B = 0.$$

This property can be extended to product of several terms, i.e.

$$\text{If} \quad A \times B \times C \times D = 0, \quad \text{then either } A = 0 \text{ or } B = 0 \text{ or } C = 0 \text{ or } D = 0$$

We thus need to express a quadratic equation in the form where **RHS is zero**, i.e.

$$ax^2 + bx + c = 0$$

The quadratic on the LHS will need to be changed into product of factors by **factorising highest common factor, or using grouping, algebraic identities or 'cross' method.**

**Class Practice 1**

**1** Solve each of the following equations.

**(a)** $5a^3 - 125a = 0$

$$5a^3 - 125a = 0$$
$$5a\left(a^2 - 5^2\right) = 0$$
$$5a(a+5)(a-5) = 0$$
$$a = 0 \text{ or } a = -5 \text{ or } a = 5$$

**(b)** $x(3x-1) = 2$

$$x(3x-1) = 2$$
$$3x^2 - x - 2 = 0$$
$$(3x+2)(x-1) = 0$$
$$3x+2 = 0 \quad \text{or} \quad x-1 = 0$$
$$x = -\frac{2}{3} \qquad x = 1$$

**(c)** $(7-3x)(x+2) = 4$

$$(7-3x)(x+2) = 4$$
$$7x + 14 - 3x^2 - 6x = 4$$
$$-3x^2 + x + 14 = 4$$
$$3x^2 - x - 10 = 0$$
$$(3x+5)(x-2) = 0$$
$$3x+5 = 0 \quad \text{or} \quad x-2 = 0$$
$$x = -\frac{5}{3} \qquad x = 2$$

**(d)** $y^3 + y^2 + y + 1 = 0$

$$y^3 + y^2 + y + 1 = 0$$
$$y^2(y+1) + (y+1) = 0$$
$$(y+1)\left(y^2+1\right) = 0$$
$$y+1 = 0 \quad \text{or} \quad y^2 + 1 = 0$$
$$y = -1 \qquad y^2 = -1$$

(reject as $y^2$ is always positive)

**Note:**

To solve an equation involving an unknown means to find the values of the unknown that will make the equation true.

The values are referred to as **solutions** to the equations or **roots** of the equation.

If ***a*** and ***b*** are roots of a quadratic equation, then the factors of it are $(x-a)$ and $(x-b)$.

**Example** Form a quadratic equation in $x$ with the given roots

**(a)** $3, -4$ **(b)** $\frac{2}{3}, -\frac{4}{5}$

$x = 3$ or $x = -4$

$x - 3 = 0$ $x + 4 = 0$

$(x-3)(x+4) = 0$

$x^2 + 4x - 3x - 12 = 0$

$x^2 + x - 12 = 0$

$x = \frac{2}{3}$ or $x = -\frac{4}{5}$

$3x = 2$ $5x = -4$

$3x - 2 = 0$ $5x + 4 = 0$

$(3x-2)(5x+4) = 0$

$15x^2 + 12x - 10x - 8 = 0$

$15x^2 + 2x - 8 = 0$

OR $x = \frac{2}{3}$ or $x = -\frac{4}{5}$

$x - \frac{2}{3} = 0$ $x + \frac{4}{5} = 0$

$(x - \frac{2}{3})(x + \frac{4}{5}) = 0$

$x^2 + \frac{4}{5}x - \frac{2}{3}x - \frac{8}{15} = 0$

$x^2 + \frac{2}{15}x - \frac{8}{15} = 0$

## 1.2 Solving Quadratic Equations by Completing the Square

### 1.2.1 Solving Quadratic Equations of the form $(x+a)^2 = b$

**Example** Solve each of the following equations.

**(a)** $(5x-4)^2 = 81$ **(b)** $\left(\frac{1}{2} - x\right)^2 = 10$

$5x - 4 = \pm\sqrt{81}$

$5x - 4 = \pm 9$

$5x - 4 = 9$ or $5x - 4 = -9$

$5x = 13$ $5x = -5$

$x = \frac{13}{5}$ $x = -1$

$x = 2\frac{3}{5}$

$\frac{1}{2} - x = \pm\sqrt{10}$ or $\frac{1}{2} - x = -\sqrt{10}$

$\frac{1}{2} - x = \sqrt{10}$

$x = \frac{1}{2} - \sqrt{10}$

$x = -2.66$ (2dp)

$x = \frac{1}{2} + \sqrt{10}$

$x = 3.66$ (2dp)

### 1.2.2 Completing the square for quadratic expression of the form $x^2 + px$

When direct factorisation cannot be applied, it will be useful to rewrite the equation into the form $(x+p)^2 = q$ where $p$ and $q$ are real numbers.

Recall your identity: $(a+b)^2 = a^2 + 2ab + b^2$

In particular, $(x+a)^2 = x^2 + 2ax + a^2$ or $(x-a)^2 = x^2 - 2ax + a^2$

**Note: $(x+a)^2$ is considered a perfect square.**

Let us consider the expansion of $(x+3)^2$.

$(x+3)^2 =$ $x^2 + 6x + 9$

Try to arrange this into a square.

Recall that, $(x+3)^2 = (x+3)\times(x+3)$.

| $\times$ | $x$ | $3$ |
|---|---|---|
| $x$ | $x^2$ | $3x$ |
| $3$ | $3x$ | $9$ |

In general, quadratic equations of the form $(x+a)^2$ can be arranged into a multiplication frame similar to what we did above.

What about $x^2+6x$?

| $\times$ | $x$ | $3$ |
|---|---|---|
| $x$ | $x^2$ | $3x$ |
| $3$ | $3x$ | ? |

What is the number that we must add to complete the square? __**9**____

However, $x^2+6x \neq (x+3)^2$.

| $\times$ | $x$ | $3$ |
|---|---|---|
| $x$ | $x^2$ | $3x$ |
| $3$ | $3x$ | |

$\neq$

| $\times$ | $x$ | $3$ |
|---|---|---|
| $x$ | $x^2$ | $3x$ |
| $3$ | $3x$ | $9$ |

Since we need to **add** _9_ to $x^2+6x$, we must _subtract 9____ so that **the original expression does not change.**

$$x^2+6x = \underline{x^2+6x+9} - 9$$

$$= (x+3)^2 - 9$$

From this we realised that to make a quadratic expression of the form $x^2+px$ into a perfect square $(x+a)^2$ we have to **add a number, *b***.

Let us find out what is the number that we have to add.

| Quadratic expression<br>$x^2 + px$ | Number that must be added to complete the square, $b$ | $\dfrac{\text{coefficient of } x}{2} = \dfrac{p}{2}$ | Quadratic expression of the form<br>$(x+a)^2 - b$ |
|---|---|---|---|
| (a) $x^2+6x$<br>$\begin{array}{c\|cc} \times & x & 3 \\ \hline x & x^2 & 3x \\ 3 & 3x & 9 \end{array}$ | $3^2=9$ | $\frac{6}{2}=3$ | $x^2+6x$<br>$=x^2+6x+3^2-3^2$<br>$=(x+3)^2-9$ |
| (b) $x^2+4x$<br>$\begin{array}{c\|cc} \times & x & 2 \\ \hline x & x^2 & 2x \\ 2 & 2x & 4 \end{array}$ | $2^2=4$ | $\frac{4}{2}=2$ | $x^2+4x$<br>$=x^2+4x+2^2-2^2$<br>$=(x+2)^2-4$ |
| (c) $x^2+8x$<br>$\begin{array}{c\|cc} \times & x & 4 \\ \hline x & x^2 & 4x \\ 4 & 4x & 16 \end{array}$ | $4^2=16$ | $\frac{8}{2}=4$ | $x^2+8x$<br>$=x^2+8x-4^2-4^2$<br>$=(x+4)^2-16$ |
| (d) $x^2+10x$<br>$\begin{array}{c\|cc} \times & x & 5 \\ \hline x & x^2 & 5x \\ 5 & 5x & 25 \end{array}$ | $5^2=25$ | $\frac{10}{2}=5$ | $x^2+10x$<br>$=x^2+10x+5^2-5^2$<br>$=(x+5)^2-25$ |

1. What is the relationship between $b$ and $p$? $b=\left(\frac{p}{2}\right)^2$

2. What is the relationship between $a$ and $p$? $a=\frac{p}{2}$

Hence, to make $x^2+px$ a perfect square, $x^2+px=x^2+px+\left(\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2$

$$x^2+px=\left(x+\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2$$

In general, to make $x^2 + px$ into a perfect square, $\left(\frac{p}{2}\right)^2$ must be added to it, where $p$ is the coefficient of $x$.

**Example** Complete the following expressions such that they are perfect squares.

| Complete the following expressions such that they are perfect squares. | Express $x^2 + px$ in completed square form. |
|---|---|
| (1) $x^2 + 7x + \left(\frac{7}{2}\right)^2 = \left(x + \frac{7}{2}\right)^2$ | $x^2 + 7x = \left(x + \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2$ |
| (2) $x^2 - 3x + \left(\frac{3}{2}\right)^2 = \left(x - \frac{3}{2}\right)^2$ | $x^2 - 3x = \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$ |
| (3) $x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = \left(x + \frac{7}{4}\right)^2$ | $x^2 + \frac{7}{2}x = \left(x + \frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2$ |
| (4) $c^2 + \frac{14}{3}c + \left(\frac{7}{3}\right)^2 = \left(c + \frac{7}{3}\right)^2$ | $c^2 + \frac{14}{3}c = \left(c + \frac{7}{3}\right)^2 - \left(\frac{7}{3}\right)^2$ |
| (5) $v^2 - \frac{7}{8}v + \left(\frac{7}{16}\right)^2 = \left(v + \frac{7}{16}\right)^2$ | $v^2 - \frac{7}{8}v = \left(v + \frac{7}{16}\right)^2 - \left(\frac{7}{16}\right)^2$ |

Note that coefficient of $x^2$ must be 1.

### 1.2.3 Completing the square for quadratic expression of the form $x^2 + px + q$

If instead of $x^2 + px$, we are given the expression $x^2 + px + q$, we can still convert the **first two terms into perfect squares.**

**Example** Express the following in the form $(x+a)^2 + c$, where $a$ and $c$ are real numbers.

**(a)** $x^2 + 6x + 11$

$$
\begin{aligned}
& x^2 + 6x + 11 \\
&= \left[\left(x + \frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2\right] + 11 \\
&= \left[(x+3)^2 - 9\right] + 11 \\
&= (x+3)^2 + 2
\end{aligned}
$$

**(b)** $2x^2 - 8x + 3$

$$
\begin{aligned}
& 2x^2 - 8x + 3 \\
&= 2\left(x^2 - 4\right) + 3 \\
&= 2\left[\left(x - \frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right] + 3 \\
&= 2\left[(x-2)^2 - 4\right] + 3 \\
&= 2(x-2)^2 - 8 + 3 \\
&= 2(x-2)^2 - 5
\end{aligned}
$$

**In conclusion,**

$$x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

$$\begin{aligned} x^2 + px + q &= \left(x^2 + px\right) + q \\ &= \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q \end{aligned}$$

### 1.2.4 Solving Equations of the form $x^2 + px + q = 0$ by using Completing the Square

**Steps to solve a quadratic equation by completing the square**

$$x^2 + px + q = 0$$

$$\downarrow$$

$$x^2 + px = -q$$ [add $-q$ to both sides of the equation]

$$\downarrow$$

$$\left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 = -q$$ [complete the square on LHS]

$$\downarrow$$

$$\left(x + \frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q$$ [Make the perfect square term the subject]

$$\downarrow$$

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}$$ [Take square root on both sides. Remember to add plus-minus sign]

$$\downarrow$$

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$ [Make $x$ the subject]

**Example** Solve $x^2+6x+6=0$.

$$x^2+6x+6=0$$
$$x^2+6x=-6$$
$$(x+3)^2-9=-6$$
$$(x+3)^2=3$$
$$x+3=\pm\sqrt{3}$$
$$x=-3\pm\sqrt{3}$$
$$x\approx-4.73 \quad \text{or} \quad x\approx-1.27$$

**Class Practice 2**

**1** Express each of the following expressions in the form $(x+a)^2+b$.

**(a)** $x^2+9x-1$

$$x^2+9x-1$$
$$=\left[\left(x+\frac{9}{2}\right)^2-\left(\frac{9}{2}\right)^2\right]-1$$
$$=\left(x+\frac{9}{2}\right)^2-\frac{81}{4}-1$$
$$=\left(x+\frac{9}{2}\right)^2-\frac{85}{4}$$

**(b)** $x^2-1.4x$

$$x^2-1.4x$$
$$=\left(x-\frac{1.4}{2}\right)^2-\left(\frac{1.4}{2}\right)^2$$
$$=(x-0.7)^2-(0.7)^2$$
$$=(x-0.7)^2-0.49$$

**2** Solve each of the following equations.

**(a)** $x^2+3x-4=0$

$$x^2+3x-4=0$$
$$x^2+3x=4$$
$$\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2=4$$
$$\left(x+\frac{3}{2}\right)^2=\left(\frac{9}{4}\right)+4$$
$$\left(x+\frac{3}{2}\right)^2=\frac{25}{4}$$
$$\left(x+\frac{3}{2}\right)=\pm\sqrt{\frac{25}{4}}$$
$$x=-\frac{3}{2}\pm\frac{5}{2}$$
$$x=-4 \quad \text{or} \quad x=1$$

It is much faster to factorise!

**(b)** $2x^2+0.6x-1=0$

$$2x^2+0.6x-1=0$$
$$x^2+0.3x-0.5=0$$
$$x^2+0.3x=0.5$$
$$\left(x+\frac{0.3}{2}\right)^2-\left(\frac{0.3}{2}\right)^2=0.5$$
$$(x+0.15)^2=(0.15)^2+0.5$$
$$(x+0.15)^2=0.5225$$
$$(x+0.15)=\pm\sqrt{0.5225}$$
$$x=-0.15\pm\sqrt{0.5225}$$
$$x\approx-0.873 \quad \text{or} \quad x\approx0.573$$

**3** Solve $x(x-3)=5x+1$.

$$x^2-3x=5x+1$$
$$x^2-8x=1$$
$$x^2-8x+(-4)^2=1+(-4)^2$$
$$(x-4)^2=17$$

$x-4=\sqrt{17}$ or $x-4=-\sqrt{17}$

$x=4+\sqrt{17}$ $x=4-\sqrt{17}$

$x=8.12$ (3sf) $x=-0.123$ (3sf)

### 1.3 Solving Quadratic Equations by using Formula

Consider the quadratic equation $ax^2 + bx + c = 0$, where $a$, $b$ and $c$ are real constants.

By using the completing the square method, make $x$ the subject of the formula.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ [Divide by $a$ on both sides of the equation so coefficient of $x$ is 1]

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$ [Minus $\frac{c}{a}$ on both sides of the equation]

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = -\frac{c}{a}$$ [Complete the square of the LHS]

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$ [Make perfect square term the subject. $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$]

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$ [Combine fractions on RHS using LCM]

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$ [Take square root on both sides, include "$\pm$"]

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$ [Make $x$ the subject of the formula]

Hence the general formula for finding the solutions to a quadratic equation $ax^2 + bx + c = 0$ is

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

While the formula works for all cases, it is generally faster to use factorisation where possible.

**Example** Solve each of the following equations.

**(a)** $x^2+4x+1=0$

**(b)** $-3x^2-7x+9=0$

(a)

$x^2+4x+1=0 \quad (a=1, b=4, c=1)$

$$x=\frac{-4\pm\sqrt{4^2-4(1)(1)}}{2(1)}$$

$$=\frac{-4\pm\sqrt{12}}{2}$$

$$x=\frac{-4\pm2\sqrt{3}}{2}$$

$x=-2+\sqrt{3}$ or $x=-2-\sqrt{3}$

$\approx -0.268$ $\approx -3.73$

(b)

$-3x^2-7x+9=0$

$a=-3,\ b=-7,\ c=9$

$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(-3)(9)}}{2(-3)}$$

$$=\frac{7\pm\sqrt{157}}{-6}$$

$x=\frac{7+\sqrt{157}}{-6}$ or $x=\frac{7-\sqrt{157}}{-6}$

$x=-3.25$ (3sf) $x=0.922$ (3sf)

**Class Practice 3**

**1** Solve each of the following equations using the quadratic formula.

**(a)** $x(x-5)=7-2x$

**(b)** $3x-4=(4x-3)^2$

(a)

$x(x-5)=7-2x$

$x^2-5x=7-2x$

$x^2-3x-7=0$

$a=1,\ b=-3,\ c=-7$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-7)}}{2(1)}$$

$$=\frac{3\pm\sqrt{37}}{2}$$

$x=\frac{3+\sqrt{37}}{2}$ or $x=\frac{3-\sqrt{37}}{2}$

$x=4.54$ (3sf) $x=-1.54$ (3sf)

(b)

$3x-4=(4x-3)^2$

$3x-4=16x^2-24x+9$

$16x^2-27x+13=0$

$a=16,\ b=-27,\ c=13$

$$x=\frac{-(-27)\pm\sqrt{(-27)^2-4(16)(13)}}{2(16)}$$

$$=\frac{27\pm\sqrt{-103}}{32}$$

$\therefore$ There are no real roots.

### 1.4 Solving Fractional Equations that can be reduced to Quadratic Equations

Equations that contain one or more algebraic fractions are known as fractional equations.

Examples of fractional equations are $\frac{2}{5x-1}=x+2$ and $\frac{3}{x+2}=2-\frac{x-1}{x-5}$.

In this section, we will solve fractional equations that can be reduced to quadratic equations.

**Example** Solve $\frac{(x-2)(x-3)}{(x-1)(x+2)}=\frac{2}{3}$.

$$\frac{x^2-5x+6}{x^2+x-2}=\frac{2}{3}$$

$$3(x^2-5x+6)=2(x^2+x-2)$$

$$3x^2-15x+18=2x^2+2x-4$$

$$x^2-17x+22=0$$

$a=1$, $b=-17$, $c=22$

$$x=\frac{-(-17)\pm\sqrt{(-17)^2-4(1)(22)}}{2(1)}$$

$$=\frac{17\pm\sqrt{201}}{2}$$

$x=\frac{17+\sqrt{201}}{2}$ or $x=\frac{17-\sqrt{201}}{2}$

$x=15.6$ (3sf) $\quad x=1.41$ (3sf)

**Class Practice 4**

**1** Solve each of the following equations.

**(a)** $\dfrac{x-2}{5}+\dfrac{1}{2x-3}=1$

$$\frac{(x-2)(2x-3)+5}{5(2x-3)}=1$$

$$\frac{2x^2-3x-4x+6+5}{10x-15}=1$$

$$2x^2-7x+11=10x-15$$

$$2x^2-17x+26=0$$

$a=2$, $b=-17$, $c=26$

$$x=\frac{-(-17)\pm\sqrt{(-17)^2-4(2)(26)}}{2(2)}$$

$$=\frac{17\pm\sqrt{81}}{4}$$

$$=\frac{17\pm9}{4}$$

$x=\dfrac{17+9}{4}$ or $x=\dfrac{17-9}{4}$

$=6.5$ $\quad$ $=2$

or $(2x-13)(x-2)=0$

**(b)** $\dfrac{5}{x-2}=2-\dfrac{4}{(x-2)^2}$

$$\frac{4}{(x-2)^2}+\frac{5}{x-2}=2$$

$$\frac{4+5(x-2)}{(x-2)^2}=2$$

$$4+5x-10=2(x-2)^2$$

$$5x-6=2(x^2-4x+4)$$

$$5x-6=2x^2-8x+8$$

$$2x^2-13x+14=0$$

$a=2$, $b=-13$, $c=14$

$$x=\frac{-(-13)\pm\sqrt{(-13)^2-4(2)(14)}}{2(2)}$$

$$=\frac{13\pm\sqrt{57}}{4}$$

$x=\dfrac{13+\sqrt{57}}{4}$ or $x=\dfrac{13-\sqrt{57}}{4}$

$=5.14$ (3sf) $\quad$ $=1.36$ (3 sf)

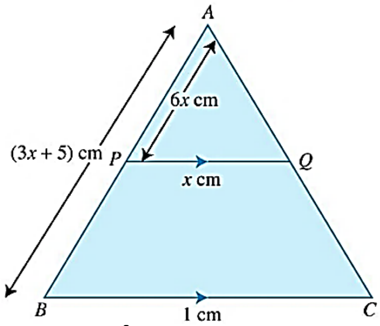
## 1.5 Applications of Quadratic Equations in Real-World Contexts

Note: In Problem Solving

- Define any unknown values/Write down necessary expressions.
- Formulate an equation.
- Proceed to solve for the unknown values (using factorization, completing the square or general formula).
- Check for validity of answers.
- Remember to conclude your answers. Answer the question specifically!

**Example**

The figure shows a triangle $ABC$ in which $AP = 6x$ cm, $AB = (3x+5)$ cm, $PQ = x$ cm and $BC = 1$ cm. $P$ and $Q$ are two points on the lines $AB$ and $AC$ respectively such that $\frac{AP}{AB} = \frac{PQ}{BC}$.




**(i)** Formulate an equation in $x$ and show that it reduces to $3x^2 - x = 0$.

$\frac{AP}{AB} = \frac{PQ}{BC}$

$\frac{6x}{3x+5} = \frac{x}{1}$

$6x = x(3x+5)$ * brackets!

$6x = 3x^2 + 5x$

$3x^2 - x = 0$ (shown)

**(ii)** Solve the equation $3x^2 - x = 0$.

$3x^2 - x = 0$

$x(3x-1) = 0$ factorise! NEVER divide by $x$! why?

$x = 0$ or $3x - 1 = 0$

$x = \frac{1}{3}$ * remember NOT to reject any solutions

**(iii)** Find the length of $PB$.

$PB = (3x+5) - 6x$

$= (5-3x)$ cm

When $x = \frac{1}{3}$,

$PB = 5 - 3(\frac{1}{3})$

$= 5 - 1$

$= 4$ cm

**Example**

In January 2009, the price of rice in Singapore was $\$x$ per kilogram. A food catering company spent an average of \$350 on rice each month.

**(i)** Write down an expression, in terms of $x$, for the average amount of rice that this food catering company ordered in January 2009.

Amount of rice ordered in Jan 2009 = $\frac{350}{x}$ kg

In January 2012, the price of each kilogram of rice had increased by 15 cents.

**(ii)** Given that the company continued to spend \$350 on rice each month, write down an expression, in terms of $x$, for the average amount of rice ordered in January 2012.

Amound of rice ordered in Jan 2012 = $\frac{350}{x+0.15}$ kg

**(iii)** If the difference in the amount of rice ordered is 30kg, formulate an equation in $x$ and show that it reduces to $20x^2+3x-35=0$.

$$\frac{350}{x}-\frac{350}{x+0.15}=30$$

$$\frac{350(x+0.15)-350x}{x(x+0.15)}=30$$

$$\frac{350x+52.5-350x}{x^2+0.15x}=30$$

$$52.5=30(x^2+0.15x)$$

$$52.5=30x^2+4.5x$$

$$30x^2+4.5x-52.5=0$$

$$x^2+0.15x-1.75=0$$

$$20x^2+3x-35=0 \quad \text{(shown)}$$

**(iv)** Hence find the price of each kilogram of rice in January 2012.

$a=20,\ b=3,\ c=-35$

$$x=\frac{-3\pm\sqrt{3^2-4(20)(-35)}}{2(20)}$$

$$=\frac{-3\pm\sqrt{2809}}{40}$$

$x=\frac{-3+53}{40}$ or $x=\frac{-3-53}{40}$

$x=1.25$ $\qquad$ $x=-1.4$

$\therefore$ Price of each kg of rice in Jan 2012

$=1.25+0.15$

$=\$1.40$.

**Class Practice 5**

**1** Rui Feng and Jun Wei represented their class in a 10 km race. They started running at the same speed of $x$ km/h. After 2 km, Rui Feng increased his speed by 1 km/h and ran the remaining distance at a constant speed of $(x+1)$ km/h. Jun Wei maintained his speed of $x$ km/h throughout the race.

**(i)** Write down an expression, in terms of $x$, for the time taken by Rui Feng to complete the race.

Time taken by Rui Feng $= \left(\frac{2}{x} + \frac{8}{x+1}\right)$ h

**(ii)** Given that Rui Feng completed the race 40 minutes earlier than Jun Wei, formulate an equation in $x$ and show that it reduces to $x^2 + x - 12 = 0$.

$$\frac{10}{x} - \left(\frac{2}{x} + \frac{8}{x+1}\right) = \frac{40}{60}$$
$$\frac{10}{x} - \frac{2}{x} - \frac{8}{x+1} = \frac{2}{3}$$
$$\frac{8}{x} - \frac{8}{x+1} = \frac{2}{3}$$
$$\frac{8(x+1) - 8x}{x(x+1)} = \frac{2}{3}$$
$$\frac{8x+8-8x}{x^2+x} = \frac{2}{3}$$
$$\frac{8}{x^2+x} = \frac{2}{3}$$
$$8(3) = 2(x^2+x)$$
$$24 = 2x^2+2x$$
$$2x^2+2x-24 = 0$$
$$x^2+x-12 = 0 \quad \text{(shown)}$$

**(iii)** Solve the equation $x^2 + x - 12 = 0$. Explain why you reject one of the answers.

$$x^2+x-12 = 0$$ * always write the equation before you factorise!
$$(x+4)(x-3) = 0$$
$x+4=0$ or $x-3=0$
$x=-4$ $\qquad$ $x=3$

$x=-4$ is rejected as speed cannot be negative.

**(iv)** Hence find the time taken by Rui Feng to complete the race, giving your answer in hours and minutes.

Time taken by Rui Feng $= \frac{2}{3} + \frac{8}{4}$
$= 2\frac{2}{3}$ h
$= 2$ h $40$ min.

**2** Two weeks before Nora went to New York for a holiday, she exchanged S$2000 into US dollars (US$) at Samy's Money Exchange at a rate of US$1 = S$$x$.

**(i)** Write down an expression, in terms of $x$, for the amount of US$ she received from Samy's Money Exchange.

Amt. of US$ received $= \dfrac{2000}{x}$

One week before her holiday, she exchanged another S$1000 into US$ at Chan's Money Exchange at a rate of US$1 = S$$(x+0.05)$.

**(ii)** Write down an expression, in terms of $x$, for the amount of US$ she received from Chan's Money Exchange.

Amt. of US$ received $= \dfrac{1000}{x+0.05}$

**(iii)** If Nora received a total of US$2370 from the two Money Exchanges, formulate an equation in $x$ and show that it reduces to $237x^2 - 288.15x - 10 = 0$.

$$\frac{2000}{x} + \frac{1000}{x+0.05} = 2370$$

$$\frac{2000(x+0.05) + 1000x}{x(x+0.05)} = 2370$$

$$\frac{2000x + 100 + 1000x}{x^2 + 0.05x} = 2370$$

$$3000x + 100 = 2370(x^2 + 0.05x)$$

$$3000x + 100 = 2370x^2 + 118.5x$$

$$2370x^2 - 2881.5x - 100 = 0$$

$$237x^2 - 288.15x - 10 = 0 \text{ (shown)}$$

**(iv)** Solve the equation $237x^2 - 288.15x - 10 = 0$, giving both your answers correct to 2 decimal places.

$$237x^2 - 288.15x - 10 = 0$$

$a = 237$, $b = -288.15$, $c = -10$

$$x = \frac{-(-288.15) \pm \sqrt{(-288.15)^2 - 4(237)(-10)}}{2(237)}$$

$$= \frac{288.15 \pm \sqrt{92510.4225}}{474}$$

$x \approx 1.2496$ or $x \approx -0.033766$

$x = 1.25$ (2dp) $\quad x = -0.03$ (2dp)

**(v)** Find the exchange rate between S$ and US$ offered by Chan's Money Exchange.

$$x + 0.05$$
$$\approx 1.2496 + 0.05$$
$$= 1.30 \text{ (2dp)}$$

$\therefore$ Exchange rate at Chan's Money Exchange is US$1 = S$1.30.

## 1.6 Sketching Graphs of Quadratic Functions

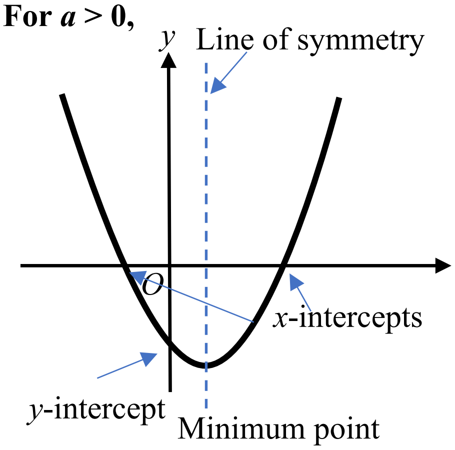
In Sec 2, you have learnt how to plot the graphs of quadratic functions $y = ax^2 + bx + c$, where $a$, $b$ and $c$ are real constants and $a \neq 0$.

In this section, we learn how to **sketch** the graph **without the graph paper** using **pencil**. In a sketch, the **key properties and critical points** of the graph **must be explicitly shown**:

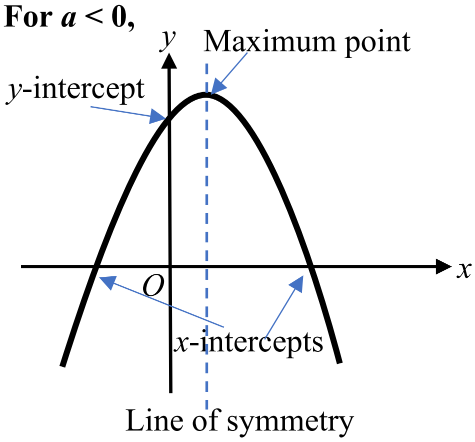
**S** – Shape (General shape, symmetry)
**I** – Axial Intercepts (Coordinates of points of intersection with $x$ and $y$ axes)
**T** – Turning points (Coordinates of maximum or minimum points)

Let's recap some features of these graphs.

**For $a > 0$,**




**For $a < 0$,**




Next, we will learn how to find the **coordinates of critical points** when a quadratic equation is expressed in different forms.

### 1.6.1 Factorised Form $y = (x-h)(x-k)$ or $y = -(x-h)(x-k)$

**Example** Sketch the graph of $y = x\left(x+\frac{3}{4}\right)$.

On the $x$-axis, $y = 0$

$$x\left(x+\frac{3}{4}\right) = 0$$

$$x = 0 \quad \text{or} \quad x = -\frac{3}{4}$$

On the $y$-axis, $x = 0$, $y = 0$.

Line of symmetry is

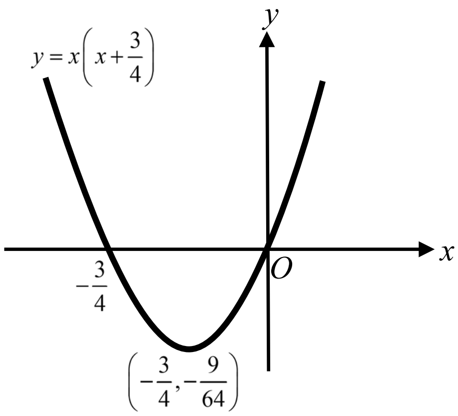
$$x = \frac{0+\left(-\frac{3}{4}\right)}{2}$$

$$x = -\frac{3}{8}$$

When $x = -\frac{3}{8}$,

$$y = \left(-\frac{3}{8}\right)^2 + \frac{3}{4}\left(-\frac{3}{8}\right) = -\frac{9}{64}$$

$\therefore$ Min point $\left(-\frac{3}{8}, -\frac{9}{64}\right)$



**Notes:**

**1.** For the equation $y=(x-h)(x-k)$, the graph

- opens **upwards**,
- cuts the $x$-axis at $(h,0)$ and $(k,0)$,
- is symmetrical about the line $x=\dfrac{h+k}{2}$, which passes through the minimum point.

**2.** For the equation $y=-(x-h)(x-k)$, the graph

- opens **downwards**,
- cuts the $x$-axis at $(h,0)$ and $(k,0)$,
- is symmetrical about the line $x=\dfrac{h+k}{2}$, which passes through the maximum point.

**Class Practice 6**

**1** Sketch the function $y=x^2-4x+3$.

$y = x^2-4x+3$
$\quad = (x-3)(x-1)$

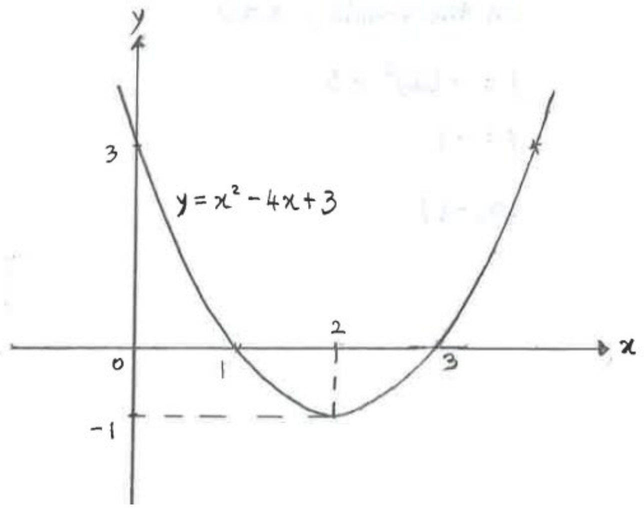
$\therefore$ $x$ intercepts are at $(3,0)$ and $(1,0)$.

Line of symmetry is
$x = \dfrac{3+1}{2}$
$x = 2$

When $x=2$, $y=(2-3)(2-1)$
$y = -1$

$\therefore$ min-pt is $(2,-1)$

When $x=0$, $y=3$



**2** Sketch the graph of $y=(3-x)(x+2)$.

On the $x$-axis, $y=0$

$(3-x)(x+2)=0$

$3-x=0$ or $x+2=0$

$x=3$ $\quad$ $x=-2$

$x=\frac{3-2}{2}$

$=\frac{1}{2}$

$\therefore$ Line of symmetry is $x=\frac{1}{2}$.

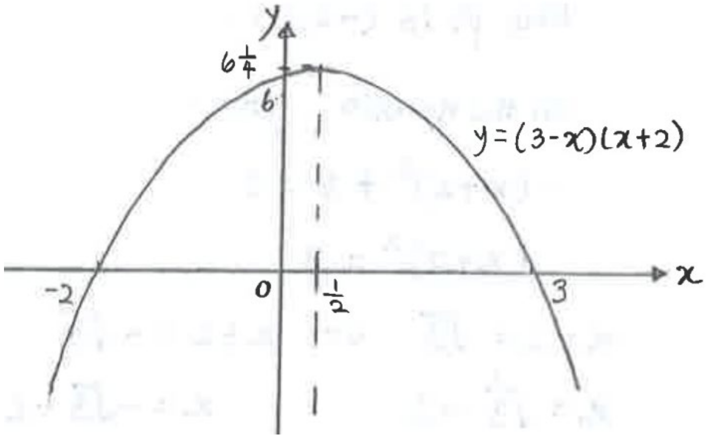
When $x=\frac{1}{2}$, $y=(3-\frac{1}{2})(\frac{1}{2}+2)$

$=6\frac{1}{4}$

$\therefore$ max. pt. is $(\frac{1}{2}, 6\frac{1}{4})$.

When $x=0$, $y=3(2)$

$=6$




### 1.6.2 Completed Square Form $y=(x-p)^2+q$ or $y=-(x-p)^2+q$.

Completed square form is most useful when we want to deduce the coordinates of the maximum or minimum point.

Since $(x-p)^2$ is a **perfect square**, it **can never be negative**.

As $(x-p)^2 \geq 0$ for all $x$ values, the minimum value of $y=(x-p)^2+q$ is when $(x-p)^2=0$.

Minimum value of $y=0+q=q$

This happens when $x=p$.

The same argument works for finding maximum value of $y=-(x-p)^2+q$

**Notes:**

**1.** For the equation $y=(x-p)^2+q$,

- the graph opens **upwards**,
- the coordinates of the **minimum** point of the graph are $(p,q)$,
- and the graph is symmetrical about the line $x=p$.

**2.** For the equation $y=-(x-p)^2+q$,

- the graph opens **downwards**,
- the coordinates of the **maximum** point of the graph are $(p,q)$,
- and the graph is symmetrical about the line $x=p$.

**Example** Sketch the graph of $y=(x-2)^2-9$. State the coordinates of the minimum point of the graph and the equation of the line of symmetry of the graph.

On the $x$-axis, $y=0$

$(x-2)^2-9=0$

$(x-2)^2=9$

$x-2=\pm 3$

$x=-1$ or $x=5$

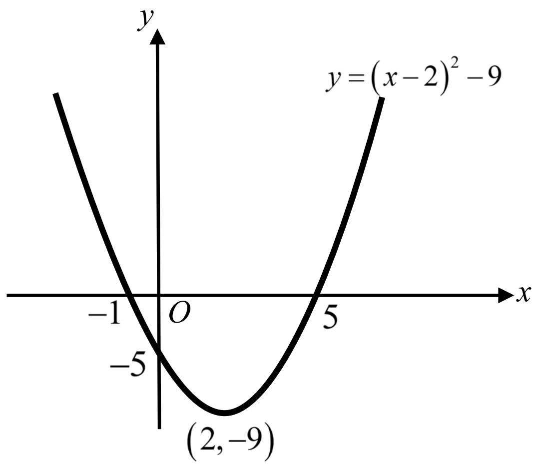
On the $y$-axis, $x=0$,

$y=(0-2)^2-9=-5$

Min point $(2,-9)$

Line of symmetry:

$x=2$




**Class Practice 7**

**1** Sketch the function $y=(x-4)^2-11$. State the equation of the line of symmetry.

min. pt is $(4,-11)$.

When $x=0$, $y=5$.

$\therefore (0,5)$

When $y=0$, $(x-4)^2-11=0$

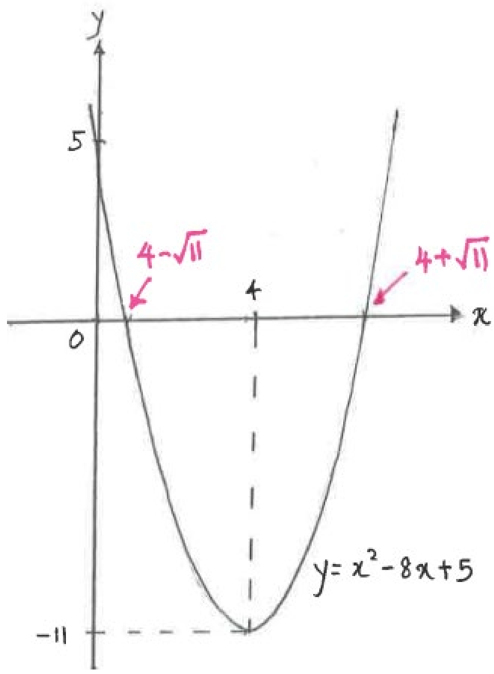
$(x-4)^2=11$

$x-4=\sqrt{11}$ or $x-4=-\sqrt{11}$

$x=4+\sqrt{11}$ $\quad$ $x=4-\sqrt{11}$

$x\approx 7.32$ $\quad$ $x\approx 0.683$

Line of symmetry is $x=4$



**2** Sketch the graph of $y=-(x+2)^2+3$, indicating the coordinates of the turning point and the point of intersection with the $y$-axis. Deduce the equation of the line of symmetry.

Max · pt is $(-2, 3)$.

on the $x$-axis, $y=0$.

$-(x+2)^2+3=0$

$(x+2)^2=3$

$x+2=\sqrt{3}$ or $x+2=-\sqrt{3}$

$x=\sqrt{3}-2$ $\quad$ $x=-\sqrt{3}-2$

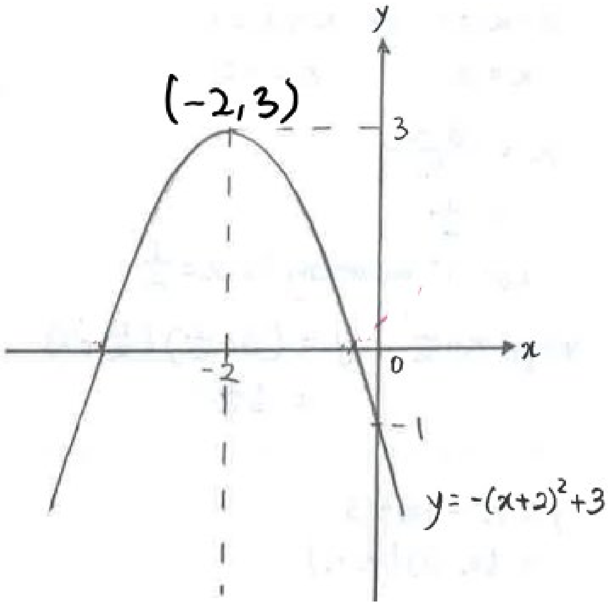
$x\approx-0{\cdot}268$ $\quad$ $x\approx-3{\cdot}73$

on the $y$-axis, $x=0$

$y=-(2)^2+3$

$y=-1$

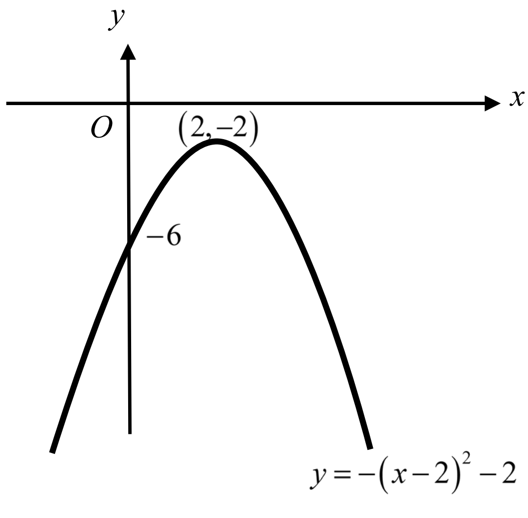
$(0,-1)$




line of symmetry is $x=-2$.

**3** Express $y=-x^2+4x-6$ in the form $y=-(x-p)^2+q$. Hence sketch the graph of $y=-x^2+4x-6$.

$$\begin{aligned}y&=-x^2+4x-6\\&=-(x^2-4x)-6\\&=-\left[(x-2)^2-2^2\right]-6\\&=-(x-2)^2+2^2-6\\&=-(x-2)^2-2\end{aligned}$$




On the $x$-axis, $y=0$

$-(x-2)^2-2=0$

$(x-2)^2=-2$

No real solution as $(x-2)^2\geq0$ for all $x$ values

On the $y$-axis, $x=0$,

$y=-(0-2)^2-2=-6$

Max point $(2,-2)$