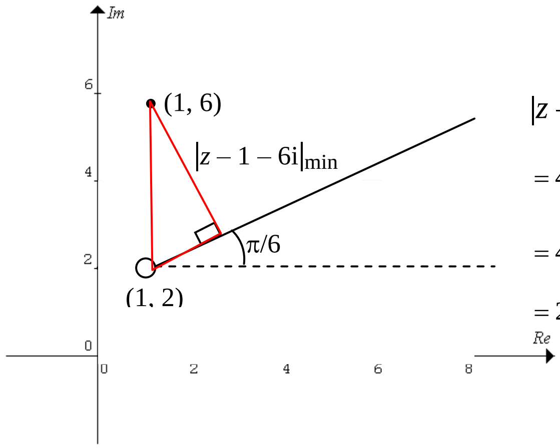
Victoria Junior College
H2 Mathematics (9740) – 2012 Preliminary Examinations Paper 2

This list of common mistakes and comments are compiled to maximise your learning through the mistakes made. Do not miss the opportunity. Go through all the details carefully so that you will not make the same mistakes in the A level exams.

1. (i) The locus of point P is the half-line from (1, 2) (excluding it), inclined at an angle of θ with the positive real axis.

(ii)



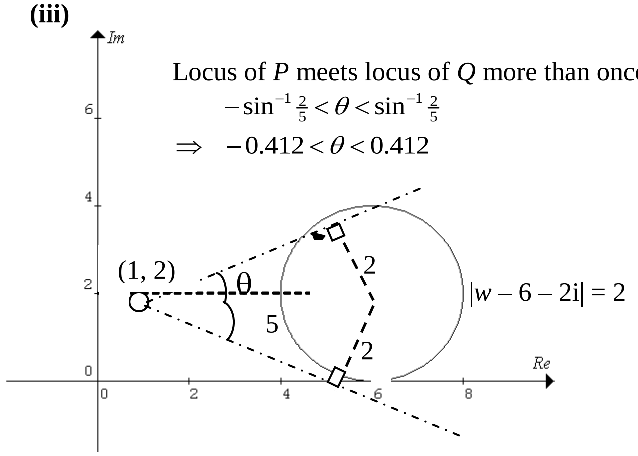
$$|z-1-6i|_{min} = 4\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = 4\sin\frac{\pi}{3} = 2\sqrt{3}$$

(iii)

Locus of P meets locus of Q more than once if

$$-\sin^{-1}\tfrac{2}{5} < \theta < \sin^{-1}\tfrac{2}{5}$$

$$\Rightarrow \quad -0.412 < \theta < 0.412$$



2. (i) $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$

$$\sum_{r=1}^{n}\left(2r+\frac{1}{r(r+1)}\right) = \sum_{r=1}^{n}2r + \sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$= 2\frac{n}{2}(1+n) + \left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= n(1+n)+1-\frac{1}{n+1}$$

(ii) $\displaystyle\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}\left(2r+\frac{1}{r(r+1)}\right)$

$$\begin{aligned} & u_1 - u_0 \\ & +u_2 - u_1 \\ & \ldots \\ & +u_n - u_{n-1} \end{aligned} = n(n+1)+1-\frac{1}{n+1}$$

$$u_n - 1 = n(n+1)+1-\frac{1}{n+1} \qquad \text{(since } u_0 = 1 \text{ as given)}$$

$$u_n = n(n+1)+2-\frac{1}{n+1} \quad \text{(shown)}$$

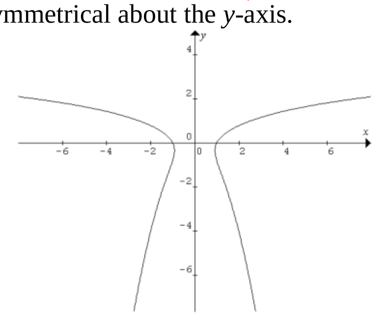
(iii) Let $k = r+1$. When $r=1$, $k=2$;
when $r = n-1$, $k = n$

$$\therefore \sum_{r=1}^{n-1}\left(2(r+1)+\frac{1}{(r+1)(r+2)}\right)$$
$$=\sum_{k=2}^{n}\left(2k+\frac{1}{k(k+1)}\right)$$
$$=\sum_{k=1}^{n}\left(2k+\frac{1}{k(k+1)}\right)-\left(2+\frac{1}{1(1+1)}\right)$$
$$=\left(n(n+1)+1-\frac{1}{n+1}\right)-\frac{5}{2}$$
$$=n(n+1)-\frac{1}{n+1}-\frac{3}{2}$$

Alternative

$$\sum_{r=1}^{n-1}\left(2(r+1)+\frac{1}{(r+1)(r+2)}\right)$$
$$= 4+6+8+\dots+2n$$
$$+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}$$
$$=\sum_{r=1}^{n}\left(2r+\frac{1}{r(r+1)}\right)-2-\frac{1}{1.2}$$
$$=n(n+1)+1-\frac{1}{n+1}-\frac{5}{2}$$
$$=n(n+1)-\frac{1}{n+1}-\frac{3}{2}$$

3. i) Since replacing x with $-x$ results in the same equation, the graph is symmetrical about the y-axis.



(ii)

Differentiate $e^{2y} = x^2 + y$ with respect to x:

$$e^{2y}.2\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2e^{2y}-1\right) = 2x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{2e^{2y}-1} \text{ (shown)}$$

(iii) The line $x = k$ is vertical, so at $x = k$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

$$2e^{2y}-1=0 \Rightarrow y = \frac{1}{2}\ln\frac{1}{2}.$$

When $y = \frac{1}{2}\ln\frac{1}{2}$,

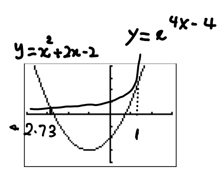
$$e^{2\left(\frac{1}{2}\ln\frac{1}{2}\right)} = k^2 + \frac{1}{2}\ln\frac{1}{2}$$
$$k^2 = \frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}\left(1+\ln 2\right)$$

Hence, $k = \pm\sqrt{\frac{1+\ln 2}{2}}$.

(iv) When $x=1$, $y=0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.

Equation of tangent : $y - 0 = 2(x-1)$
$\Rightarrow y = 2x - 2$.

Solving equations of the tangent and the curve simultaneously, we get
$e^{4x-4} = x^2 + 2x - 2$.



From GC, $x = 1$ (rejected as this gives the previous point)
or $x = -2.73$ (3 s.f.)

4. (i)

$$\left.\begin{aligned} 2x-4y+z&=6 \\ x+y-z&=6 \end{aligned}\right\} \text{solve simultaneously using GC,} \quad \begin{aligned} x&=5+\tfrac{1}{2}z \\ y&=1+\tfrac{1}{2}z \\ z&=0+z \end{aligned}$$

So the equation of the line l is $\mathbf{r}=\begin{pmatrix}5\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\2\end{pmatrix},\ \lambda\in\mathbb{R}$.

(ii) Given that the system of equations of the 3 planes has infinitely many solutions, it implies that line l lies on the plane p_3.

So $\begin{pmatrix}5+\lambda\\1+\lambda\\2\lambda\end{pmatrix}\cdot\begin{pmatrix}a\\-2\\-1\end{pmatrix}=b$ for all real values of λ.

$$5a+\lambda a-2-2\lambda-2\lambda=b$$

$$\Rightarrow 5a+\lambda(a-4)-2=b$$

This is satisfied when a = 4 and therefore b = 18.

<u>OR</u>

$l\perp$ normal of $p_3\Rightarrow\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}a\\-2\\-1\end{pmatrix}=0\Rightarrow a=4$.

(5,1,0) lies in $p_3\Rightarrow\begin{pmatrix}5\\1\\0\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\-1\end{pmatrix}=b\Rightarrow b=18$

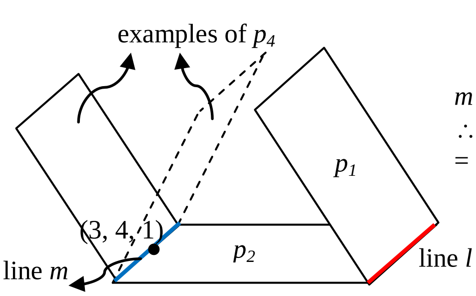
(iii) plane is perpendicular to p_1 and $p_2\Rightarrow$ it is perpendicular to l.

$\therefore$ normal to the plane is $\begin{pmatrix}1\\1\\2\end{pmatrix}$.

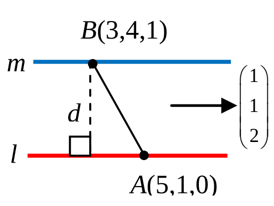
Equation of the plane is $\mathbf{r}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}$

$$\therefore\mathbf{r}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=4.$$

(iv) p_1, p_2 and p_4 have no common point:



m is parallel to l.
$\therefore$ distance between m and l
= distance from (3,4,1) to l.



$$d=\frac{\left|\overrightarrow{AB}\times\begin{pmatrix}1\\1\\2\end{pmatrix}\right|}{\sqrt{1+1+2^2}}=\frac{\left|\begin{pmatrix}-2\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}\right|}{\sqrt{6}}=\frac{\left|\begin{pmatrix}5\\5\\-5\end{pmatrix}\right|}{\sqrt{6}}$$

$$=\frac{5\sqrt{3}}{\sqrt{6}}=\frac{5}{\sqrt{2}}\text{ units}$$

5. **(i)** Let μ cm be the population mean increase in heights of orchid plants in the nursery.

$H_0 : \mu = 1.34$

$H_1 : \mu > 1.34$

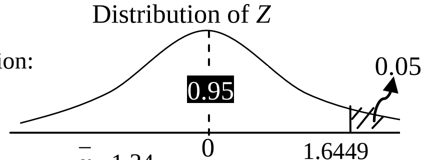
As n = 55 is large, by Central Limit Theorem, $\overline{X}$ is approximately normal. $\therefore$ no assumption about the increase in heights is needed.

(ii) Level of significance: 5%

Test-statistic: When H_0 is true, $Z = \dfrac{\overline{X} - 1.34}{\frac{0.24}{\sqrt{55}}} \sim N(0,1)$

Distribution of Z

Rejection Region:



To NOT reject H_0, $z = \dfrac{\overline{x} - 1.34}{\frac{0.24}{\sqrt{55}}} < 1.6449$

$$\Rightarrow \overline{x} < 1.34 + 1.6449\left(\frac{0.24}{\sqrt{55}}\right) \approx 1.3932$$

$\therefore \left\{\overline{x} \in \mathbb{R} \; : \; \overline{x} < 1.39\right\}$

(iii) There is a 5% probability of the test concluding that there is a significant improvement in the mean increase in heights of orchids when there is not.

6 (a) Due to time and resource constraints, the committee would prefer to poll a sample instead of all members.

Obtain a numbered name list of the club members. Use a random number generator to choose an integer from 1 to 10. Choose the member corresponding to that number on the name list followed by every 10th member subsequently until 100 members are selected.

If there are cyclical patterns in the name list, the survey might be biased.

(b) Case 1: Adam & Bernice are together : $9! \times 2! = 725760$

Case 2: Adam & Bernice separated by exactly 1 person:

$$8! \times {}^8C_1 \times 2! = 645120$$

Required number of ways = $10! - 725760 - 645120 = 2257920$

7. Let X be the number of free gifts claimed per week. $X \sim \text{Po}(8)$.

(i) Let n denote the number of gifts kept in stock at the start of the week. For $P(X \le n) \ge 0.95$, from the GC:

n	$P(X \le n)$
12	0.9362
13	0.96582

$\therefore$ the least number of free gifts needed is 13.

7. (ii) Let Y be the number of free gifts claimed in m weeks.

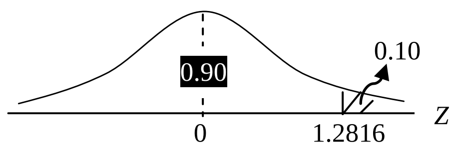
$$Y \sim \text{Po}(8m)$$

*****If $8m > 10$, $Y \sim \text{N}(8m, 8m)$ approximately.

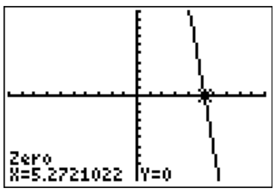
$\text{P}(Y > 50) \le 0.1$

Using continuity correction, $\text{P}(Y > 50.5) \le 0.1$

Standardising, we have $\text{P}\left(Z > \dfrac{50.5 - 8m}{\sqrt{8m}}\right) \le 0.10$.



$y = 50.5 - 8x - 1.2816\sqrt{8x}$



$\dfrac{50.5 - 8m}{\sqrt{8m}} \ge 1.2816$

$\Rightarrow 50.5 - 8m - 1.2816\sqrt{8m} \ge 0$

$\therefore$ largest integer value of $m = 5$.

*for Normal approximation to Poisson distribution, $m > 5/4$.

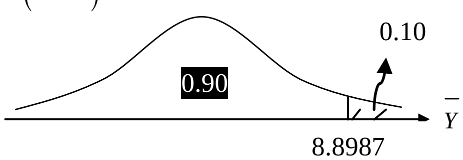
8. Let X and Y denote the waiting times, in minutes, at the fitting room for a man and a woman respectively. Given

$\text{E}(X) = 4.2$, $\text{Var}(X) = 1.6^2$ & $\text{E}(Y) = 8.5$, $\text{Var}(Y) = 2.2^2$

(i) Let $\overline{Y} = \dfrac{Y_1 + Y_2 + \cdots + Y_{50}}{50}$. By Central Limit Theorem,

$\overline{Y} \sim \text{N}\left(8.5, \dfrac{2.2^2}{50}\right)$ approximately.

For $\text{P}(\overline{Y} > k) \le 0.10$,



$k \ge 8.8987$ $\therefore$ least integer value of $k = 9$

Given $X \sim \text{N}(4.2, 1.6^2)$, $Y \sim \text{N}(8.5, 2.2^2)$,

$T = X_1 + ... + X_6 - 3Y$

(ii) $\therefore T \sim N\left(6 \times 4.2 - 3 \times 8.5,\ 6 \times 1.6^2 + 9 \times 2.2^2\right)$

$\Longrightarrow T \sim N(-0.3, 58.92)$

$\text{P}(|T| > 4) = \text{P}(T > 4) + \text{P}(T < -4)$

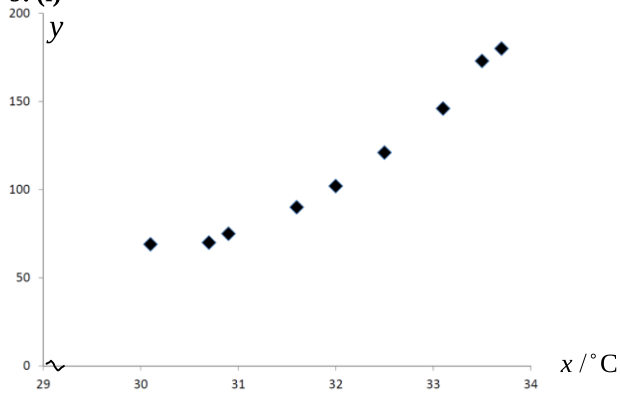
$= 0.28767 + 0.31489 = 0.603$ (to 3 s.f.)

Assumption: The waiting times for the 6 men and 1 woman are all independent of one another.

(iii) Let W be the number of woman out of 4 with waiting time more than 8.5mins. $W \sim \text{B}(4, 0.5)$

$\text{P}(W \ge 3) = 1 - \text{P}(W \le 2) = 0.3125$

9. (i)



(ii) $r = 0.971$ (to 3 s.f.)

The magnitude is close to 1, suggesting a strong positive linear correlation between the daily temperature and number of ice cream cones sold.

(iii)

(a) Using the regression line of y on x, $y = -934.47 + 32.753x$,
when $x = 33.3$, $y = 156$ (to 3 s.f.).
Since $x = 33.3$ is within the data range of x and there is a strong linear correlation between x and y, this estimate is a reliable one.

(b) Yes, the owner has reasons to suspect dishonesty. The large difference between the reported value and the reliable estimate suggests a high possibility of dishonest reporting.

(iv) For the data $\ln y$ and x, $r = 0.989$ (3s.f).
Since this value is closer to 1 compared to the earlier value of 0.971, $\ln y = a + bx$ is the better model.
Using $\ln y = 0.29042x - 4.6245$ and let initial values be x_0, y_0:

$$\ln y_0 = 0.29042x_0 - 4.6245$$

$$\ln y_1 = 0.29042(x_0 + 2) - 4.6245$$

$$= \ln y_0 + 0.58085$$

$$y_1 = y_0 e^{0.58085}$$

Percentage increase in number of ice cream cones

$$= \frac{y_1 - y_0}{y_0} \times 100\% = \frac{y_0 e^{0.58085} - y_0}{y_0} \times 100\% = 78.8\% \text{(3s.f.)}$$

10. (a) $n = 6$:

(i)

$$\text{P}\left(\text{same number on both dice in 1 trial}\right) = \frac{1}{6}$$

$$\text{P}\left(\text{score is 3 in 1 trial}\right)$$

$$= \text{P}\left(\text{one die shows 3}\right) \times \text{P}\left(\text{one die shows less than 3}\right) \times 2!$$

$$= \frac{1}{6} \times \frac{2}{6} \times 2$$

$$= \frac{1}{9}$$

$$\text{P(score is 3 with 2 trials)} = \frac{1}{6} \times \frac{1}{9}$$

$$= \frac{1}{54} \text{ (shown)}$$

Q10 (ii)

P(score is 3 with 1 trial) $=\frac{1}{9}$

P(score is 3 with 2 trials)$=\frac{1}{6}\times\frac{1}{9}$

P(score is 3 with 3 trials)$=\frac{1}{6}\times\frac{1}{6}\times\frac{1}{9}=\left(\frac{1}{6}\right)^2\times\frac{1}{9}$

$\vdots$

P(score is 3) = P(score is 3 with 1 trial) + P(score is 3 with 2 trials) + $\cdots$

$$=\frac{1}{9}+\left(\frac{1}{6}\right)\frac{1}{9}+\left(\frac{1}{6}\right)^2\frac{1}{9}+\cdots$$

$$=\frac{1}{9}\left(\frac{1}{1-\frac{1}{6}}\right) \text{ (sum to infinity of GP)}$$

$$=\frac{2}{15}$$

(iii)

$$\text{P(exactly 4 trials | score is 3)} = \frac{\text{P(score is 3 with 4 trials)}}{\text{P(score is 3)}}$$

$$=\frac{\left(\frac{1}{6}\right)^3\frac{1}{9}}{\frac{2}{15}}=\frac{5}{1296}$$

(b) n-sided dice

$\text{P}\left(\text{same number on both dice in 1 trial}\right)=\frac{1}{n}$

P(score is 3 with 1 trial) $=\frac{1}{n}\left(\frac{2}{n}\right)\times 2=\frac{4}{n^2}$

P(score is 3) = P(score is 3 with 1 trial) + P(score is 3 with 2 trials) + $\cdots$

$$=\frac{4}{n^2}+\left(\frac{1}{n}\right)\frac{4}{n^2}+\left(\frac{1}{n}\right)^2\frac{4}{n^2}+\cdots$$

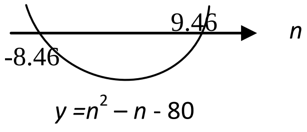
$$=\frac{4}{n^2}\left(\frac{1}{1-\frac{1}{n}}\right) \text{ (sum to infinity of GP)}$$

$$=\frac{4}{n(n-1)} \text{ (shown)}$$

$\therefore a=4, b=1$

$$\frac{4}{n\left(n-1\right)}\geq 0.05$$

$n^2-n-80\leq 0$



$4\leq n\leq 9.46$

$\therefore$ greatest n = 9

Alternative:

Sketch $Y_1=\frac{4}{x(x-1)}$

X	Y1
6	.13333
7	.09524
8	.07143
9	.05556
10	.04444
11	.03636
12	.0303

X=9

From GC, greatest n = 9