2012 NYJC H2 Math Preliminary exam Paper 2 solutions

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<tr><td>1</td><td>

(a) $zw = (1+\mathrm{i}a)(-b-\mathrm{i}) = -b-\mathrm{i}-ab\mathrm{i}+a = 6-11\mathrm{i}$

$$a-b=6$$
$$1+ab=11$$

$\Rightarrow 1+a(a-6)=11$

$$a^2-6a-10=0$$

$a = \dfrac{6\pm\sqrt{36+40}}{2} = \dfrac{6\pm\sqrt{76}}{2} = 3\pm\sqrt{19}$

Since $a > 0$, $a = 3+\sqrt{19}$, $b = -3+\sqrt{19}$

(b) $|u| = 2, \arg u = -\dfrac{2}{3}\pi$ $\quad |v| = 5, \arg v = \dfrac{3}{4}\pi$

$$\left|\frac{v}{u^2}\right| = \frac{5}{4}$$

$$\arg\left(\frac{v}{u^2}\right) = \arg v - 2\arg u = \frac{3}{4}\pi + \frac{4}{3}\pi = \frac{25}{12}\pi$$

$$\therefore \arg\left(\frac{v}{u^2}\right) = \frac{1}{12}\pi$$

$\left(\dfrac{v}{u^2}\right)^n = \left(\dfrac{5}{4}\right)^n\left(\cos\dfrac{n\pi}{12} + \mathrm{i}\sin\dfrac{n\pi}{12}\right)$

Since $\left(\dfrac{v}{u^2}\right)^n$ is imaginary, $\cos\dfrac{n\pi}{12} = 0$

$\Rightarrow \dfrac{n\pi}{12} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

Since Im part of $\left(\dfrac{v}{u^2}\right)^n$ is negative, $\sin\dfrac{n\pi}{12} < 0$

$\Rightarrow \dfrac{n\pi}{12} = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \ldots$

Hence, smallest n = 18

Alternatively,

Since $\left(\dfrac{v}{u^2}\right)^n$ is imaginary and negative,

$\dfrac{n\pi}{12} = -\dfrac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$

$n = -6+24k$

To get smallest positive n,

let k =1.

Hence, smallest n = 18.

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(i) Clearly f is 1-1 on the domain $(-1,1)$ since every horizontal line $y = k$ ($k \in \mathbb{R}$) cuts the graph at most once. So f^{-1} exists.

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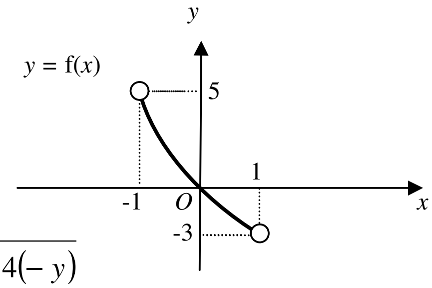


Let $y = x^2 - 4x$.

Rearranging, we have $x^2 - 4x - y = 0$.

Solving for x, we obtain $x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(-y)}}{2}$

$= 2 \pm \sqrt{4+y}$.

Since $|x| < 1$, $x = 2 - \sqrt{4+y}$ as $2 + \sqrt{4+y} \geq 2$.

So $f^{-1}(y) = x = 2 - \sqrt{4+y}$.

Hence, $f^{-1} : x \to 2 - \sqrt{4+x}$, $x \in \mathbb{R}$, $-3 < x < 5$.

(ii) Since $R_f = (-3,5) \subset \mathbb{R} \backslash \{-3\} = D_g$, gf exists.

(iii) $gf(x) = g(x^2 - 4x) = 1 + \dfrac{1}{(x^2 - 4x) + 3}$

$= 1 + \dfrac{1}{x^2 - 4x + 3}$

(iv) $h'(x) = -\dfrac{2x-4}{(x^2 - 4x + 3)^2} = \dfrac{4-2x}{(x^2 - 4x + 3)^2}$

Given $-1 < x < 1 \Rightarrow -2 < -2x < 2 \Rightarrow 2 < 4 - 2x < 6$.

Since $4 - 2x > 0$ *and* $(x^2 - 4x + 3)^2 > 0$, $h'(x) > 0$ and so h is increasing on D_h.

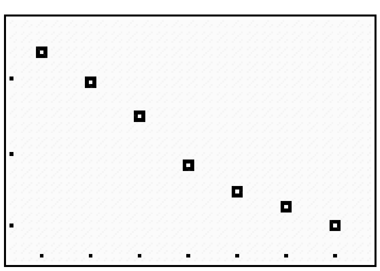
Note that $h(-1) = 1 + \dfrac{1}{1+4+3} = \dfrac{9}{8}$ and

$h(x) \to \infty$ as $x \to 1^-$.

Hence $R_h = \left(\dfrac{9}{8}, \infty\right)$ or $(1.125, \infty)$.

3	$x^2 + y^2 = \frac{1}{4}$. Let V be the volume of the silo. $V = \pi x^2 y$ $= \pi(\frac{1}{4} - y^2)y$ $= \pi(\frac{1}{4}y - y^3)$ km^3 $\frac{dV}{dy} = \pi(\frac{1}{4} - 3y^2) = 0$ $\Rightarrow y = \frac{1}{2\sqrt{3}}$ (reject $-\frac{1}{2\sqrt{3}}$) $\frac{d^2V}{dy^2} = -6\pi y < 0$ when $y = \frac{1}{2\sqrt{3}}$. maximum $V = \pi(\frac{1}{4} - \frac{1}{12})\frac{1}{2\sqrt{3}} = \pi\frac{1}{12\sqrt{3}}$ or $\pi\frac{\sqrt{3}}{36}$ km^3 Let V_d be the volume of the dome. $V_d = \frac{2}{3}\pi r^3 \Rightarrow \frac{dV_d}{dt} = 2\pi r^2\frac{dr}{dt}$ $\Rightarrow 0.75 = = 2\pi r^2(\frac{3}{4\pi})$ $\Rightarrow r^2 = 0.5 \Rightarrow r = 1/\sqrt{2}$ km $A = 2\pi r^2$ $\frac{dA}{dt} = 4\pi r\frac{dr}{dt}$ $= 4\pi(\frac{1}{\sqrt{2}})\frac{3}{4\pi}$ $= \frac{3\sqrt{2}}{2}$ km^{2}s^{-1}
4	(a) Area R $= 2\ln 2 - \int_{-1}^{1}\ln(2-\lvert x\rvert)\,dx$ or $2\ln 2 - 2\int_{0}^{1}\ln(2-\lvert x\rvert)\,dx$

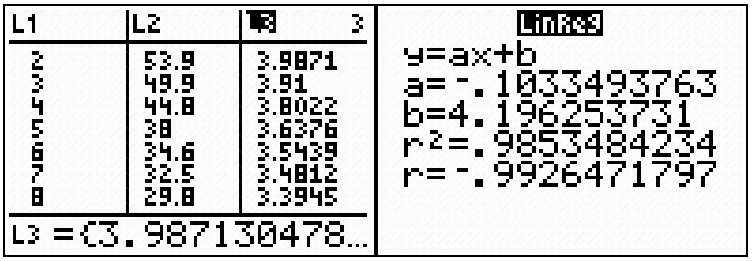
	$= 2\ln 2 - 2\int_0^1 \ln(2-x)\ \mathrm{d}x$ since $x > 0$ $= 2\ln 2 - 2\left\{\left[x\ln(2-x)\right]_0^1 + \int_0^1 \frac{x}{2-x}\mathrm{d}x\right\}$ $= 2\ln 2 - 2\left\{0 + \int_0^1 -1 + \frac{2}{2-x}\mathrm{d}x\right\}$ $= 2\ln 2 - 2\left[-x - 2\ln(2-x)\right]_0^1$ $= 2\ln 2 - 2(-1 + 2\ln 2)$ $= 2 - 2\ln 2$ (b) Volume needed $= \pi\int_0^1 y^2\ \mathrm{d}x$ When $x = 0$, $0 = (1+t)^{\frac{2}{3}} \Rightarrow t = -1$ When $x = 1$, $1 = (1+t)^{\frac{2}{3}} \Rightarrow 1 + t = \pm 1$ $t = -2$ or 0 (rejected out of range) $x = (1+t)^{\frac{2}{3}}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{3}(1+t)^{-\frac{1}{3}}$ Thus, volume needed $= \pi\int_{-1}^{-2}\left[\ln\left(t^2\right)\right]^2\ \frac{2}{3}(1+t)^{-\frac{1}{3}}\mathrm{d}t$ $= 1.80$ units^3
5	Either stratified sampling or quota sampling method can be accepted as appropriate methods. Description - Stratified– i) states that sampling frame is obtained listing the participants according to the types of educational institutions that they come from ii)states that using simple random sampling, 35, 25, 15 and 5 participants are selected

	from each of the strata (primary, secondary, JC, poly) respectively Advantage – participants from all the 4 institutions are represented proportionately Responses can be analysed by the institutions that participants come from Disadvantage – relatively inconvenient to carry out Quota – States/sets the quota to be selected from each institution type, such that they add up to 80 in total (e.g. 30 primary, 20 secondary, 20 JC, 10 poly) Selects from a list the participants for the survey, in accordance with the quota set above. Advantage – easy to choose the 80 participants Disadvantage – non-random method, so results obtained may be biased (e.g. the list may cluster participants from the same school)
6	(i) ${}^{6}C_{2}\times{}^{5}C_{4}=75$ ways (ii) Number of ways if at least one of the sisters are included = number of ways without restriction – number of ways if none of the sisters is included $={}^{11}C_{6}-{}^{8}C_{6}=434$ **Or** ${}^{3}C_{1}\times{}^{8}C_{5}+{}^{3}C_{2}\times{}^{8}C_{4}+{}^{3}C_{3}\times{}^{8}C_{3}=434$ (iii) Select a man to be between the 2 sisters and group the 3 of them as one unit and arrange 4 units round a table ${}^{3}C_{1}\times 3!\times 2=36$ (iv)First arrange the other 4 persons round the table. There are 4 ways to insert the sisters. $3!\times 4=24$ or ${}^{4}C_{2}\times 2!\times 2!=24$
7	(i)By G.C., we obtain a scatter plot of the diagram as follows: 

(ii)Although from the scatter diagram, it seems that there is a negative linear relationship between y and x within the given range. However, we note that the trade-in value of the car cannot continue to decrease linearly to become a negative value eventually. Thus, a linear model may not be appropriate.

(iii)Under a quadratic model, the trade-in value of the car would eventually increase over time, which would not make sense.

(iv)By G.C., we transform the y values as follows:



Thus, we have $\ln y = -0.103x + 4.196$.

When $x = 5.5$, $y = 37.631$. Thus, the trade-in value is \$37631.

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8 (i) P(first red bead is obtained on or before the 5th draw)

$$= \frac{2}{5} + \frac{3}{5}\cdot\frac{2}{5} + \left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) \text{ or } \sum_{r=0}^{4}\left(\frac{3}{5}\right)^r\left(\frac{2}{5}\right)$$

$= 0.922$

Or 1 – P(no red on first 5 draws) = $1 - \left(\frac{3}{5}\right)^5 = 0.922$

(ii) P(obtaining a first green bead on the 8th draw given that no green bead has been obtained after 5 draws) = P(red on 6th and 7th draws and green on 8th draw)= $\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right) = 0.096$

(iii) P(exactly r draws are required for beads of both colours to be obtained)

$$= \left(\frac{2}{5}\right)^{r-1}\left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)^{r-1}\left(\frac{2}{5}\right)$$

	$= \left(\frac{2}{5}\right)^{r-2}\left(\frac{6}{25}\right)+\left(\frac{3}{5}\right)^{r-2}\left(\frac{6}{25}\right)$ $= \left(\frac{6}{25}\right)\left[\left(\frac{2}{5}\right)^{r-2}+\left(\frac{3}{5}\right)^{r-2}\right]$, where $r = 2,3,4,\ldots$ (iv) P(first obtaining beads of different colours after 5 or more draws) $=\left(\frac{6}{25}\right)\left[\left(\left(\frac{2}{5}\right)^{3}+\left(\frac{3}{5}\right)^{3}\right)+\left(\left(\frac{2}{5}\right)^{4}+\left(\frac{3}{5}\right)^{4}\right)+\ldots\right]$ $=\left(\frac{6}{25}\right)\left[\frac{\left(\frac{2}{5}\right)^3}{1-\frac{2}{5}}+\frac{\left(\frac{3}{5}\right)^3}{1-\frac{3}{5}}\right]$ $=0.155$ Or P(first obtaining beads of different colours after 5 or more draws) = P(obtaining same colour in the first 4 draws) = P(first 4 red beads) + P(first 4 green beads) $= \left(\frac{2}{5}\right)^{4}+\left(\frac{3}{5}\right)^{4} = 0.155$
9	Assume that the mass of sodium in a box follows a normal distribution. To test $H_0: \mu = 1183$ $H_1: \mu \neq 1183$ at 5% level of significance Under H_0, $T = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(14)$ Critical Region: Reject H_0 if p-value < 0.05 Calculation: $\bar{x} = \frac{\sum(x-1180)}{15}+1180 = \frac{17.5}{15}+1180 = 1181.166667$, $s^2 = \frac{1}{n-1}\left(\sum(x-1180)^2 - \frac{\left(\sum(x-1180)\right)^2}{n}\right)$, $= \frac{1}{14}\left(190.5-\frac{17.5^2}{15}\right) = 12.14880952$ n=15 $t_{calc} = \frac{1181.166667-1183}{\left(\sqrt{\frac{12.14880952}{15}}\right)}$

	$p-\text{value}=0.0610$ Conclusion: Since p-value = 0.0610 > 0.05, we do not reject H_0. There is insufficient evidence, at 5% level of significance, to say that the mean mass of sodium in a box of cheese rings has changed. (ii) To test $H_0:\mu=\mu_0$ $H_1:\mu>\mu_0$ at 5% level of significance Under H_0, $Z=\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim \text{N}(0,1)$ Critical Region: Reject H_0 if $z_{\text{calc}}>1.64485$ Calculation: $\overline{x}=1181.166667$, $\sigma^2=6.8^2$, n=15 $z_{\text{cal}}=\dfrac{\overline{x}-\mu_0}{6.8/\sqrt{15}}$ Since H_0 is rejected, $z_{\text{cal}}>1.64485$, ie. $\mu_0<1178.28$ The assumption of the masses being normal distribution is still necessary as the sample size is small/Cannot use CLT
10	(i) Let X be the number of minutes after 6:30 am that Peter takes to reach the bus stop. $X\sim N(0,10^2)$ Prob Req'd = $P(X>20)=0.0228$ (ii) Let Y be the number of minutes for the bus journey. $Y\sim N(45,20^2)$ Prob.Req'd = $P(X>20)+P(X\le 10)\cdot P(Y>50)+P(10<X\le 20)\cdot P(Y>40)$ $\approx 0.44174\approx 0.442$ Let W be the number of days in a five days week in which Peter will be late for school. $W\sim B(5,0.442)$. Prob. Req'd = $P(W\ge 2)$ $=1-P(W\le 1)$ $=0.732$

	$E(W)=2.21,\ Var(W)=1.23318$ By Central Limit Theorem, $\overline{W} \sim N(2.21, 0.0308295)$ approx. $\text{Prob. Req'd} = P(\overline{W} \geq 2)$ ≈ 0.884
11	Let X be the number of calls reporting on a lost card in 1 hour, $X \sim \text{Po}(18)$ (i)Required prob = $P(X > 12) = 1 - P(X \leq 12)$ = 0.908(3308) (ii)Let Y be the total number of calls received in 15 minutes, $Y \sim \text{Po}(4.5+3+6)$ Required prob = $P(Y \leq 12) = 0.409$ (iii)Let W be the number of one-hour periods where more than 12 calls reporting a lost card are received in an hour, out of 24 one-hour periods. $W \sim$ B(24, part I answer) Required prob = $P(W \geq 20) = 1 - P(W \leq 19)$ = 0.937 binomial, 0.936947, sensitive to i) answer used in 3 dp; 0.9361448 if 0.908 is used (iv)Let U be number of nuisance calls received in 1 hour, and V be number of calls requesting for increase in credit limit in 1 hour. $U \sim \text{Po}(24)$, $V \sim \text{Po}(12)$, and $X \sim \text{Po}(18)$ (from part i)) can be approximated by normal distributions $U \sim N(24, 24)$, $V \sim N(12,12)$ and $X \sim N(18,18)$ respectively since the Poisson means are more than 10. Hence, $U - (X+V)$ approximately $N(24 - (12+18), 24+12+18)$ i.e. $N(-6, 54)$ Required prob = $P(U > X + V) = P(U - (X+V) > 0)$ = 0.188 (with c.c.) or 0.207 (without c.c.) (v)Assumptions Number of phone calls received of the various types are independent Phone calls of the various types are mutually exclusive Not accepted – enough telephone operators to receive the calls (because the poisson distn is already on calls “received”)