Anderson Junior College
Preliminary Examination 2011
H2 Mathematics Paper 1

<u>Answer ALL questions.</u>

1. Find $\int x\cos^{-1} x^2 \, \mathrm{d}x$. [3]

2. The usual selling price of a T-box game console, a Kinect sensor and a game DVD is \$499 in total. During the Great Singapore Sale, two companies, P and Q, offered the following discounts to their customers:

Company	Discounts given for each item			Total price after the discount
	T-box game console	Kinect sensor	Game DVD	
P	10%	15%	10%	\$439.15
Q	5%	25%	20%	\$426.30

The employees from Company P are offered a further 5% discount on the original price of both the Kinect sensor and the Game DVD. Determine if this additional discount will make it more attractive for the employees to purchase all the 3 items from their own company than from Company Q. [4]

3. Solve the inequality $\dfrac{(x+a)(x-b)}{(x)(x-c)^2} < 0$, where $a, b, c \in \mathbb{R}^{+}$ and $0 < c < b < a$. Hence, find the range of values of x which satisfy $\dfrac{(\ln x + a)(\ln x - b)}{(\ln x)(c - \ln x)^2} < 0$. [5]

4. (i) By using the substitution $t = \tan x$, show that $\int \dfrac{1}{1+\sin^2 x} \, \mathrm{d}x = \dfrac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\tan x\right) + c$. [3]

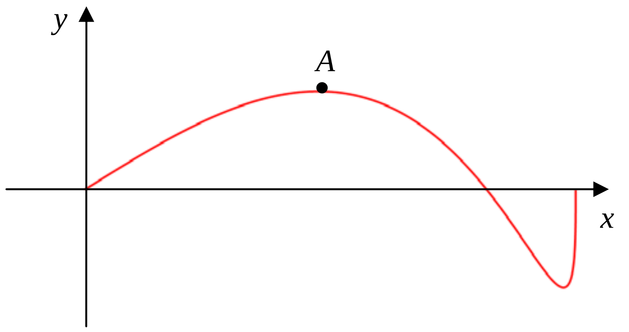
(ii) Find the exact volume of revolution when the region bounded by the curve $y = 2 + \sqrt{\dfrac{1}{1+\sin^2 x}}$, the lines $x = \dfrac{\pi}{4}$, $y = 2$ and the y-axis is rotated completely about the line $y = 2$. [3]

5. (a) Find $\int \sin 2x \cos x \, dx$. [2]

(b) The diagram below shows the curve defined by the parametric equations

$x = \sin t + t$,

$y = \sin 2t$ for $0 \le t \le \pi$.



The curve has a maximum point at A.

(i) Find the equation of the tangent to the curve at the point A. [3]

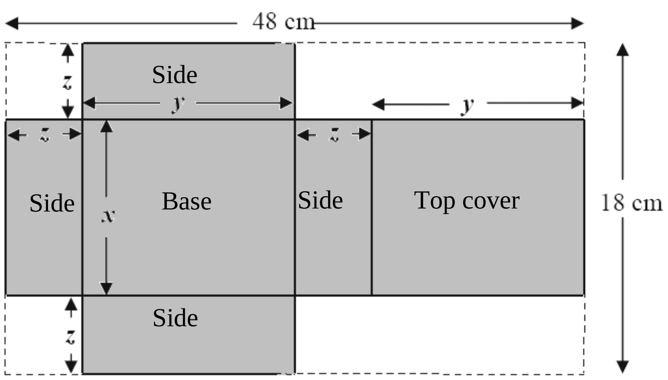
(ii) Find the exact area bounded by the curve, the tangent at A and the line $x = \pi$. [3]

6. Given that $y = \dfrac{1}{(2+\sin 2x)}$, show that $\dfrac{d^2y}{dx^2} = \dfrac{2}{y}\left(\dfrac{dy}{dx}\right)^2 + 4y^2 \sin 2x$. [3]

(i) Hence find the Maclaurin's series for y up to and including the term in x^3. [3]

(ii) By considering the standard series for sin x, verify that the series obtained in (i) is correct. [2]

7.

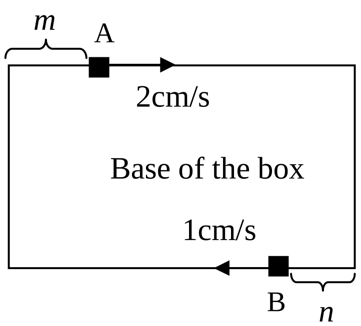


The above 48 cm by 18 cm plastic sheet is cut out to form a shape represented by the shaded region. The shape is folded into a box with width x, length y and height z as shown in the diagram.

(i) Express the volume of the box, V, in terms of y. [2]

(ii) Using differentiation, find the maximum possible volume of the box. [3]

Two small robots of negligible dimensions, A and B, are initially placed on the edge of the base of the box where m = 0 cm and n = 0 cm respectively as shown in the diagram. Robot A starts to travel along the length of the base with speed 2 cm/s. One second later, Robot B starts to travel in the opposite direction along the length of the base with speed 1 cm/s.



(iii) Given that the base of the box has length 20 cm and breadth 10 cm, find the rate of change of the distance between the two robots when n = 4 cm. [4]

8. The curve C has the equation $y = \dfrac{ax^2}{x+a}$ where a is a negative constant.

(i) Show that the curve C has two stationary points for all negative values of a. [2]

(ii) Sketch the curve C, showing clearly all the asymptotes, axial intercepts and turning points. [3]

(iii) Using the sketch in (ii), find the range of values of k for which $x^4 = \left(k - x^2\right)\left(x-1\right)^2$ has exactly two roots. [3]

9. The equations of two lines are given as follows:

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$l_2 : \mathbf{r} = \mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R}$$

(i) Show that l_1 and l_2 are skew lines. [3]

(ii) If Z is the midpoint between any point on l_1 and any point on l_2, show that the locus of Z is a plane. Write down the equation of this plane p in scalar product form. [3]

(iii) S is a point on l_1. The point S' is the image of point S reflected in plane p. If S' is also a point on l_2, find the coordinates of S. [2]

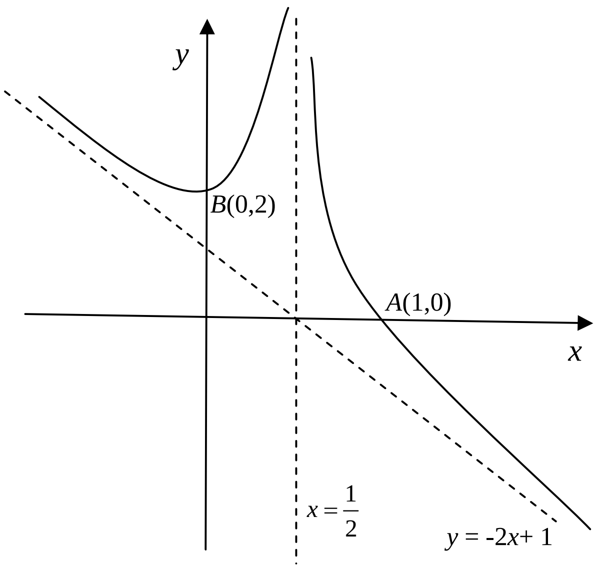
10. The sketch below shows the graph of $y = \mathrm{f}(x)$. The curve passes through the point A (1,0) and has a minimum point at B(0,2). The equation of the asymptotes are $y = -2x + 1$ and $x = \frac{1}{2}$.

On separate diagrams, sketch the following graphs indicating the points corresponding to A, B and asymptotes where necessary.

(i) $y = \mathrm{f}\left(\left|x + \frac{1}{2}\right|\right)$ [3]

(ii) $y = \frac{1}{\mathrm{f}(x)}$ [3]

(iii) $y = \mathrm{f}'(2x)$ [3]



11. A sequence $u_1, u_2, u_3, ...$ is defined by the recurrence relation $u_n = \frac{n}{n-1}u_{n-1} + \frac{1}{n-1}$,

$n \geq 2,\ n \in \mathbb{Z}$ and $u_1 = a$ where a is a constant.

(i) Use the method of mathematical induction to show that $u_n = (a+1)n - 1$ for all positive integers n. [3]

(ii) If a =1, show that $\sum_{n=2}^{N} \frac{1}{u_n u_{n-1}} = \frac{1}{2}\left(1 - \frac{1}{2N-1}\right)$. [2]

Hence, find $\sum_{n=2}^{N} \frac{1}{(2n+9)(2n+7)}$. [3]

12. The sum of the first n terms of a series, S_n, is given by

$S_n = \frac{1}{a}\left[1 - (a-1)^n\right]$, where a is a constant and $a \neq 1$, $n \in \mathbb{Z}^+$

Obtain an expression for the n^{th} term of the series, T_n and prove that S_n is a geometric series. [3]

If the sequence $\{T_n\}$ is now grouped as follows:

$(T_1),\ (T_2, T_3, T_4),\ (T_5, T_6, T_7, T_8, T_9),\ \ldots$

where each subsequent bracket has 2 terms more than the previous bracket, find

(i) the total number of terms in the first n brackets. [2]

(ii) the middle term of the 11$^{\text{th}}$ bracket in terms of a. [2]

(iii) the range of values of a for the sum to infinity of the series to exist.

Hence, find the least value of n for the sum of all the terms in the first n brackets to be within 0.1% of the sum to infinity of the series when $a = \frac{39}{20}$. [4]

13. (a) (i) The complex numbers p and q satisfy the simultaneous equations

$$p^* + 10i = qi + 5$$

$$|p|^2 - q - 5 + 2i = 0.$$

Given that Im(p) < 0, find p in the Cartesian form. [3]

Hence find the values of n for which p^{2n} is purely imaginary. [2]

(ii) The complex number w is such that $\arg\left(\frac{w}{p} - p^*\right) = -\frac{\pi}{2}$ and $w + w^* = -2$. Find w. [4]

(b) Find the roots of the equation $\left(z - \sqrt{2}\right)^6 = 8$, giving your answer in the form

$R\cos\theta\, e^{i\theta}$ where $R > 0$ and $-\pi < \theta \le \pi$. [3]

Describe the curve on which all these roots lie. [1]

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