RAFFLES INSTITUTION
2022 Year 6 H2 Mathematics Common Test
Questions and Solutions with comments

2024 Y6 H2 Math Timed Practice Revision Paper 4 Solution

1 The curve C has equation $y = \dfrac{x^2 - 2x + 4}{x - b}$, $x \neq b$.

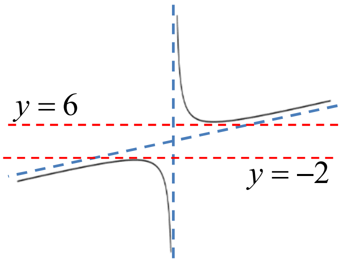
(i) Given that y cannot take values between -2 and 6, show that b = 2 using an algebraic method. [3]

(ii) Describe a sequence of 2 transformations that transform the graph of C onto the graph of $y = \dfrac{x^2 + 5x + 4}{x}$, $x \neq 0$. [3]

(i)
[3]

Method 1

Since y cannot take values between -2 and 6, then the graph of this rational function will have 2 stationary points, with minimum point at $y = 6$ and maximum point at $y = -2$.



Consider

$$\frac{x^2 - 2x + 4}{x - b} = -2 \quad \text{and} \quad \frac{x^2 - 2x + 4}{x - b} = 6$$

$$x^2 + 4 - 2b = 0 \qquad x^2 - 8x + 4 + 6b = 0$$

Since $y = -2$ and $y = 6$ are tangents to $y = \dfrac{x^2 - 2x + 4}{x - b}$, then the discriminant of the above 2 quadratic equations is equal to 0.

$$-4(4 - 2b) = 0 \quad \text{and} \quad 64 - 4(4 + 6b) = 0$$

$$b = 2 \qquad b = 2$$

Thus, b = 2.

Method 2

$$y = \frac{x^2 - 2x + 4}{x - b}$$

$$yx - by = x^2 - 2x + 4$$

$$x^2 + x(-2 - y) + (4 + by) = 0$$

$$\text{Discriminant} = (-2 - y)^2 - 4(1)(4 + by)$$

$$= y^2 + y(4 - 4b) - 12$$

Since y cannot take values between -2 and 6, then $-2 < y < 6$ is the solution for $y^2 + y(4 - 4b) - 12 < 0$.

We can rewrite $-2 < y < 6$ as $\left(y + 2\right)\left(y - 6\right) < 0$

$$y^2 - 4y - 12 < 0$$

Comparing coefficients of y, b = 2.

(ii) **[3]**	We can rewrite the 2 equations as $y=\frac{x^2-2x+4}{x-2}=x+\frac{4}{x-2}$ and $y=\frac{x^2+5x+4}{x}=x+5+\frac{4}{x}$ $y=x+\frac{4}{x-2}\xrightarrow{\text{Replace } x \text{ by } x+2}y=x+2+\frac{4}{x}$ $\xrightarrow{\text{Replace } y \text{ by } y-3}y-3=x+2+\frac{4}{x}$ $\Rightarrow y=x+5+\frac{4}{x}$ The sequence of transformation is 1) Translate C in the negative x-direction by 2 units to get $y=x+2+\frac{4}{x}$, 2) followed by a translation of 3 units in the positive y-direction.	Note that for such question, you need to **describe** (in words) the transformation used. Stating "Replace x by x+2" is <u>not</u> a description, but is part of your <u>working</u>. Need to use the proper wordings for describing transformations. In this question, "translate by k units positive/negative x/y-direction". It is mentioned by the Cambridge report that "along the x-axis", "in the x-axis" or "on the x-axis" are not accepted.

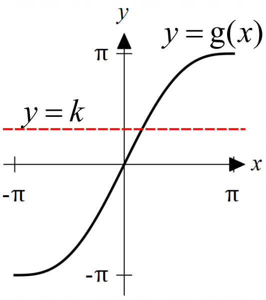
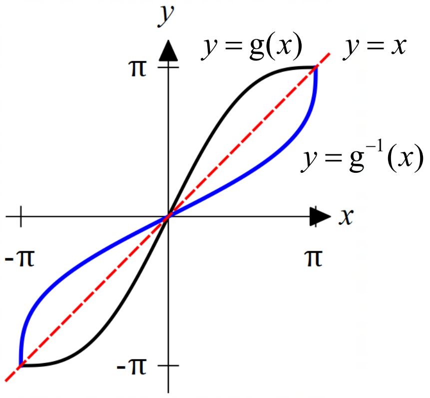
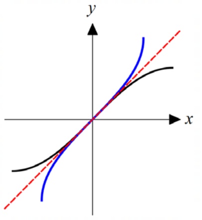
2 The function g is defined by $\mathrm{g}: x \to \sin x + x$, for $x \in \mathbb{R}$, $-\pi \le x \le \pi$.

(i) Show that $\mathrm{g}'(x) > 0$ for $-\pi < x < \pi$. Hence, or otherwise, show that g has an inverse. [2]

(ii) Sketch on the same diagram the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$. [3]

(iii) Show that the composite function g^2 exists. [1]

(iv) Find the exact solutions of the equation $\mathrm{g}^2(x) = x$. [2]

(i)	$\mathrm{g}(x) = \sin x + x$ $\mathrm{g}'(x) = \cos x + 1$ For $-\pi < x < \pi$, $-1 < \cos x \le 1$ $0 < \cos x + 1 \le 2$ $0 < \mathrm{g}'(x) \le 2$ Thus, $\mathrm{g}'(x) > 0$ for $-\pi < x < \pi$. Since g is a strictly increasing function on $-\pi \le x \le \pi$, and so g is a one-one function. Hence, g has an inverse. **Alternatively**, Any horizontal line $y = k$, where $k \in [-\pi, \pi]$, cuts the graph of $y = \mathrm{g}(x)$ at one and only one point. Thus, g is a one-one function and g^{-1} exists.  	Common mistake: $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = -\cos x$ Quite a few wrote • $k \in \mathbb{R}$ This not correct. For example, consider the line $y = 10$, the horizontal line does cut the graph of $y = \mathrm{g}(x)$ at any point. • $k \in \mathrm{R}_\mathrm{g}$ Need to state the range of g explicitly.
(ii)	 	 There are a few scripts who drew the above graph. Note that • $\mathrm{g}'(0) = 2 >$ grad of line $y = x$ • can use the GC to draw the graph of $y = \mathrm{g}(x)$ and $y = x$ on the same diagram • the x- and y- scale should be the same
(iii)	$\mathrm{R}_\mathrm{g} = [-\pi, \pi]$, $\mathrm{D}_\mathrm{g} = [-\pi, \pi]$ Since $\mathrm{R}_\mathrm{g} \subseteq \mathrm{D}_\mathrm{g}$, g^2 exists.	Some use the wrong symbol/notation: • $\mathrm{R}_\mathrm{g} \in [-\pi, \pi]$ (should be $=$) • $\mathrm{R}_\mathrm{g}[-\pi, \pi]$ (missing $=$) • $\mathrm{R}_\mathrm{g} \in \mathrm{D}_\mathrm{g}$

(iv)	$g^2(x) = x$ $g^{-1}\left(g^2(x)\right) = g^{-1}(x)$ $g(x) = g^{-1}(x)$ From the graphs of $y = g(x)$ and $y = g^{-1}(x)$ in **(ii)**, they intersect at $x = -\pi,\ 0,\ \pi$ The solutions are $x = -\pi,\ 0,\ \pi$	

3 **(i)** Prove by the method of differences that $\sum_{r=1}^{N}\frac{1}{\sqrt{r}+\sqrt{r+1}}=\sqrt{N+1}-1.$ [3]

(ii) Find $\sum_{r=4}^{N-1}\frac{1}{\sqrt{r}+\sqrt{r+1}}$ in term of N. [2]

(iii) Using part **(i)**, show that $\sum_{r=1}^{N}\frac{1}{\sqrt{r}}>2\left(\sqrt{N+1}-1\right)$ [2]

(i)	$\sum_{r=1}^{N}\frac{1}{\sqrt{r}+\sqrt{r+1}}$ $=\sum_{r=1}^{N}\frac{1}{\sqrt{r}+\sqrt{r+1}}\times\frac{\sqrt{r}-\sqrt{r+1}}{\sqrt{r}-\sqrt{r+1}}$ $=\sum_{r=1}^{N}\frac{\sqrt{r}-\sqrt{r+1}}{r-(r+1)}$ $=\sum_{r=1}^{N}\left(\sqrt{r+1}-\sqrt{r}\right)$ $=\cancel{\sqrt{2}}-1$ $+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}$ $+\cancel{\sqrt{4}}-\cancel{\sqrt{3}}$ . . . $+\cancel{\sqrt{N}}-\cancel{\sqrt{N-1}}$ $+\sqrt{N+1}-\cancel{\sqrt{N}}$ $=\sqrt{N+1}-1$	Most are able to rewrite the term by rationalizing the denominator, and obtain a difference of two related terms.
(ii)	$\sum_{r=4}^{N-1}\frac{1}{\sqrt{r}+\sqrt{r+1}}=\sum_{r=1}^{N-1}\frac{1}{\sqrt{r}+\sqrt{r+1}}-\sum_{r=1}^{3}\frac{1}{\sqrt{r}+\sqrt{r+1}}$ $=\sqrt{N}-1-(\sqrt{4}-1)$ $=\sqrt{N}-2$	
(iii)	For $r\geq 1$, $\sqrt{r}+\sqrt{r+1}>\sqrt{r}+\sqrt{r}$ $\frac{1}{\sqrt{r}+\sqrt{r+1}}<\frac{1}{\sqrt{r}+\sqrt{r}}$ $\sum_{r=1}^{N}\frac{1}{\sqrt{r}+\sqrt{r+1}}<\sum_{r=1}^{N}\frac{1}{\sqrt{r}+\sqrt{r}}$ $\sqrt{N+1}-1<\frac{1}{2}\sum_{r=1}^{N}\frac{1}{\sqrt{r}}$ $\sum_{r=1}^{N}\frac{1}{\sqrt{r}}>2\left(\sqrt{N+1}-1\right)$	

4 A curve C has parametric equations

$$x = e^t \cos t, \quad y = e^{-t} \sin t.$$

(i) Find the equation of the tangent to C at the point with parameter p. Give your answer in the form $y = mx + c$. [5]

(ii) The tangent meets the x-axis at point A and the y-axis at point B. Find, in terms of p, the area of the triangle OAB, where O is the origin. [3]

(i)	$x = e^t \cos t$ $\quad$ $y = e^{-t} \sin t$ $\frac{dx}{dt} = e^t(-\sin t) + e^t(\cos t)$ $\quad$ $\frac{dy}{dt} = e^{-t}(\cos t) - e^{-t}(\sin t)$ $= e^t(\cos t - \sin t)$ $\quad$ $= e^{-t}(\cos t - \sin t)$ $\frac{dy}{dx} = e^{-2t}$ Equation of tangent: $y - e^{-p}\sin p = e^{-2p}(x - e^p \cos p)$ $y = e^{-2p}x + e^{-p}\sin p - e^{-p}\cos p$ $y = e^{-2p}x + e^{-p}(\sin p - \cos p)$	Need to use product rule to differentiate x and y with respect to t.
(ii)	At A, $y = 0$ and $x = \frac{-e^{-p}(\sin p - \cos p)}{e^{-2p}} = -e^p(\sin p - \cos p)$ At B, $x = 0$ and $y = e^{-p}(\sin p - \cos p)$ Area of triangle OAB $= \frac{1}{2} \times \left\|-e^p(\sin p - \cos p)\right\| \times \left\|e^{-p}(\sin p - \cos p)\right\|$ $= \frac{1}{2}\left\|\sin p - \cos p\right\|^2$ $= \frac{1}{2}(\sin p - \cos p)^2$	Quite a number of solutions gave the answer as $-\frac{1}{2}(\sin p - \cos p)^2$, and not realizing that this expression is always <u>negative</u>. Actually either OA or OB is negative, so there is a need to find the absolute value for the lengths when finding the area.

5 The complex number z satisfies the equation $\mathrm{f}(z)=0$, where

$$\mathrm{f}(z)=\left(1-a\mathrm{i}\right)z^2-2\mathrm{i}z-10-20\mathrm{i}$$

and a is a real number. It is given that one root is of the form $1+b\mathrm{i}$, where b is an integer .

(i) Show that b satisfies the equation

$$b^4-2b^3+12b^2-42b-9=0.$$ [3]

(ii) Hence find a and b, and the other root of $\mathrm{f}(z)=0$. [4]

(i) **[3]**	$\left(1-a\mathrm{i}\right)\left(1+b\mathrm{i}\right)^2-2\mathrm{i}\left(1+b\mathrm{i}\right)-10-20\mathrm{i}=0$ $\left(1-a\mathrm{i}\right)\left(1+2b\mathrm{i}-b^2\right)-2\mathrm{i}+2b-10-20\mathrm{i}=0$ $1+2b\mathrm{i}-b^2-a\mathrm{i}+2ab+ab^2\mathrm{i}-2i+2b-10-20\mathrm{i}=0$ $\left(-b^2+2ab+2b-9\right)+\mathrm{i}\left(2b-a+ab^2-22\right)=0$ Comparing Real part: $-b^2+2ab+2b-9=0$ $a=\dfrac{9-2b+b^2}{2b}$ ------(1) Imaginary part: $2b-a+ab^2-22=0$ -----(2) Subst (1) into (2), $2b-\dfrac{9-2b+b^2}{2b}+\left(\dfrac{9-2b+b^2}{2}\right)b-22=0$ $4b^2-9+2b-b^2+\left(9-2b+b^2\right)b^2-44b=0$ $b^4-2b^3+12b^2-42b-9=0$	Quite a few are careless in simplifying the equation. There are a few who wrote that $1-b\mathrm{i}$ (conjugate pair) is a root of $\mathrm{f}(z)=0$. However, not all the coefficients of $\mathrm{f}(z)=0$ are real numbers. Thus, the theorem cannot be used.
(ii) **[4]**	Solving the quartic equation in **(i)** using GC and since b is an integer, we get $b=3$. Subst $b=3$ into eqn (1), we get $a=2$. **<u>Method 1</u>** Let the other root be z_0 . $\left(1-2\mathrm{i}\right)z^2-2\mathrm{i}z-10-20\mathrm{i}=\left(1-2\mathrm{i}\right)\left[z-\left(1+3\mathrm{i}\right)\right]\left(z-z_0\right)$ Comparing the constant term, $-10-20\mathrm{i}=\left(1-2\mathrm{i}\right)\left(1+3\mathrm{i}\right)z_0$ $z_0=\dfrac{-10-20\mathrm{i}}{\left(1-2\mathrm{i}\right)\left(1+3\mathrm{i}\right)}=-1.8-2.6\mathrm{i}$, using GC $\therefore$ The other root is $-1.8-2.6\mathrm{i}$	There are a few who wrote $\left(1-2\mathrm{i}\right)z^2-2\mathrm{i}z-10-20\mathrm{i}=$ $\left[z-\left(1+3\mathrm{i}\right)\right]\left[\left(1-2\mathrm{i}\right)z-z_0\right]$ This is not wrong, but do note that z_0 is **<u>not</u>** the root of the equation $\mathrm{f}(z)=0$. The root is going to be $\dfrac{z_0}{1-2\mathrm{i}}$, in this case. There are quite a few who are able to get **full marks** for **(ii)**, although not able to do **(i)**. This is commendable. • Make use of the quartic eqn in **(i)** and use GC to solve for b. • The substitute $z=1+b\mathrm{i}$ into $\mathrm{f}(z)=0$ to find the value of a (use GC)

	Method 2 $(1-2\mathrm{i})z^2-2\mathrm{i}z-10-20\mathrm{i}=0$ $z=\frac{2\mathrm{i}\pm\sqrt{-4-4(1-2\mathrm{i})(-10-20\mathrm{i})}}{2(1-2\mathrm{i})}$ $=\frac{\mathrm{i}\pm 7}{1-2\mathrm{i}}$, using GC $=1+3\mathrm{i}$ or $-\frac{9}{5}-\frac{13}{5}\mathrm{i}$ The other root is $-\frac{9}{5}-\frac{13}{5}\mathrm{i}$.	

6 The molecular geometry is the three dimensional arrangement of atoms in a molecule. Fig. 1 shows the molecular geometry of a particular molecule which is of a tetrahedral form with one central atom lying inside the tetrahedron and 4 other atoms at the vertices of the tetrahedron. The points *A, B, C* and *D* represent the 4 vertices of the tetrahedron and are connected to the central atom, which is denoted by the point *E*, forming 4 pairs of bonding atoms *AE*, *BE*, *CE* and *DE*.

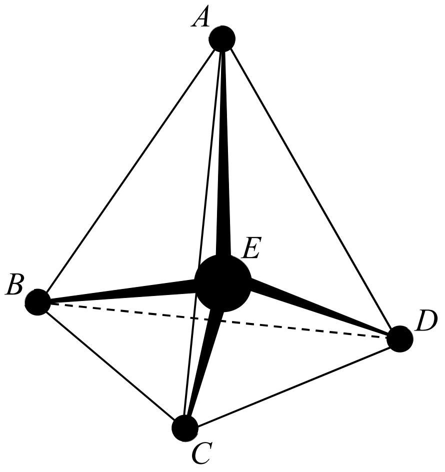


Fig.1

The coordinates of *A*, *B*, *C*, *D* and *E* are given by $(2,3,4)$, $(0,0,3)$, $(3,0,0)$, $(0,4,0)$ and $(1,1,2)$ respectively. The plane *p* passes through the points *A*, *B* and *C*. The line *l* passes through the points *D* and *E*.

(i) Find a cartesian equation of *p*. [4]

(ii) Find a vector equation of *l* and the coordinates of the point of intersection of *l* and *p*. [4]

(iii) Hence, find the shortest distance from point *E* to *p*. [2]

The bond angle is the angle between any two pairs of bonding atoms, measured in degrees.

(iv) Find the bond angle between bonding atoms *AE* and *CE*. [2]

(i) **[4]**	**<u>Method 1</u>** $\overrightarrow{AB}=\begin{pmatrix}0\\0\\3\end{pmatrix}-\begin{pmatrix}2\\3\\4\end{pmatrix}=\begin{pmatrix}-2\\-3\\-1\end{pmatrix}$; $\overrightarrow{BC}=\begin{pmatrix}3\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}3\\0\\-3\end{pmatrix}$ Since $\begin{pmatrix}-2\\-3\\-1\end{pmatrix}\times\begin{pmatrix}3\\0\\-3\end{pmatrix}=\begin{pmatrix}9\\-9\\9\end{pmatrix}$, a normal vector of *p* is $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.	

	$\begin{pmatrix}1\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\3\end{pmatrix}=3$ $\therefore$ cartesian equation of p is $x-y+z=3$ **Method 2** Let the cartesian equation of p be $ax+by+cz+d=0$. Since p passes through A, B and C, then $2a+3b+4c+d=0$ $0a+0b+3c+d=0$ $3a+0b+0c+d=0$ Using GC, we get $a=-\frac{1}{3}d,\ b=\frac{1}{3}d,\ c=-\frac{1}{3}d,\ d\in\mathbb{R}$. Thus, $-\frac{1}{3}dx+\frac{1}{3}dy-\frac{1}{3}dz+d=0\Rightarrow x-y+z=3$ (by choosing a nonzero value of d) $\therefore$ cartesian equation of p is $x-y+z=3$	Note that the question asks for the **Cartesian** equation.
(ii) **[4]**	$\overrightarrow{DE}=\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}0\\4\\0\end{pmatrix}=\begin{pmatrix}1\\-3\\2\end{pmatrix}$. So, $l:\mathbf{r}=\begin{pmatrix}0\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\2\end{pmatrix},\quad\lambda\in\mathbb{R}$ $p:\mathbf{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=3$ Substitute eqn (l) into eqn (p): $\left[\begin{pmatrix}0\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\2\end{pmatrix}\right]\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=3$ $-4+6\lambda=3$ $\lambda=\frac{7}{6}$ The coordinates of the point of intersection are $\left(\frac{7}{6},\frac{1}{2},\frac{7}{3}\right)$	There are a few who did not write the equation of the line appropriately: • Missing **r** = • Missing $\lambda\in\mathbb{R}$ Note that the question asks for the **coordinates** of the point of intersection.

(iii) **[2]**	Shortest distance $= \frac{\left\lvert\left[\begin{pmatrix}\frac{7}{6}\\ \frac{1}{2}\\ \frac{7}{3}\end{pmatrix}-\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}\right]\bullet\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}\right\rvert} = \left\lvert\begin{pmatrix}\frac{1}{6}\\ -\frac{1}{2}\\ \frac{1}{3}\end{pmatrix}\bullet\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}\right\rvert / \sqrt{3}$ $=\left(\frac{1}{6}+\frac{1}{2}+\frac{1}{3}\right)/\sqrt{3}=\frac{1}{\sqrt{3}}$ units	Note that the question state **"Hence"**, so there is a need to use the previous part answer to do this part of the question.
(iv) **[2]**	$\overrightarrow{AE}=\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}-\begin{pmatrix}2\\ 3\\ 4\end{pmatrix}=\begin{pmatrix}-1\\ -2\\ -2\end{pmatrix}$ $\overrightarrow{CE}=\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}-\begin{pmatrix}3\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}-2\\ 1\\ 2\end{pmatrix}$ $\cos\angle AEC=\frac{\begin{pmatrix}-1\\ -2\\ -2\end{pmatrix}\bullet\begin{pmatrix}-2\\ 1\\ 2\end{pmatrix}}{\left\lvert\begin{pmatrix}-1\\ -2\\ -2\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}-2\\ 1\\ 2\end{pmatrix}\right\rvert}=\frac{-4}{3(3)}=-\frac{4}{9}$ $\angle AEC=116.4^\circ$ (1 d.p.)	It is stated in the cover page that "Give non-exact numerical answers correct to 1 decimal place in the case of angles in degrees", some corrected the answer to 3 s.f. Quite a few did this: $\cos\angle AEC=\frac{\left\lvert\begin{pmatrix}-1\\ -2\\ -2\end{pmatrix}\bullet\begin{pmatrix}-2\\ 1\\ 2\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-1\\ -2\\ -2\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}-2\\ 1\\ 2\end{pmatrix}\right\rvert}=\frac{4}{9}$ $\angle AEC=63.6^\circ$ (1 d.p.) Note that the bond angle, in this question, is $\measuredangle AEC$, which not necessarily has to be an acute angle. When finding the angle between 2 vectors, it is important to make sure that both vectors are "leaving from"or "meeting at" the same point (point *E*, in this question). See Chap 4B: Vectors I notes page 8. Thus, should use either $\overrightarrow{AE}\bullet\overrightarrow{CE}$ or $\overrightarrow{EA}\bullet\overrightarrow{EC}$, and not $\overrightarrow{AE}\bullet\overrightarrow{EC}$.

7 **(a)** It is given that $x\frac{\mathrm{d}y}{\mathrm{d}x}=y+6xy$.

(i) Using the substitution $y=v^2x$, show that the differential equation can be transformed to $\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{f}(v)$, where the function $\mathrm{f}(v)$ is to be found. [3]

(ii) Hence solve the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x}=y+6xy$ to find y in terms of x. [2]

(b) A simple mathematical model for the spread of epidemics assumes that the number of people, x, infected with an infectious disease changes at a rate proportional to the product of the number of people infected with the disease and the number of people who are not yet infected by the disease, with k being the constant of proportionality.

(i) Assuming that the population remains constant at P, write down a differential equation involving x, t, P and k. [1]

(ii) Solve the differential equation and show that $x=\frac{Px_0}{x_0+(P-x_0)e^{-kPt}}$, where x_0 is the number of infected people at time $t=0$. [5]

(iii) What happens to the number of people infected with the disease if this situation continues after many years ? [1]

(a) **(i)** **[3]**	$y=v^2x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=2vx\frac{\mathrm{d}v}{\mathrm{d}x}+v^2$ Thus, $x\frac{\mathrm{d}y}{\mathrm{d}x}=y+6xy$ $x\left(2vx\frac{\mathrm{d}v}{\mathrm{d}x}+v^2\right)=v^2x+6x\left(v^2x\right)$ $2vx^2\frac{\mathrm{d}v}{\mathrm{d}x}=6v^2x^2$ $\frac{\mathrm{d}v}{\mathrm{d}x}=3v$	Some wrote $\frac{\mathrm{d}y}{\mathrm{d}x}=v^2$. Note that v^2 is not a constant, so there is a need to use product rule when differentiating y.
(a) **(ii)** **[2]**	$\frac{\mathrm{d}v}{\mathrm{d}x}=3v$ $\int\frac{1}{v}\mathrm{d}v=\int 3\,\mathrm{d}x$ $\ln\lvert v\rvert=3x+c$ $\lvert v\rvert=\mathrm{e}^{3x+c}$ $v^2=\mathrm{e}^{6x+2c}$ $\frac{y}{x}=\mathrm{e}^{6x+2c}\Rightarrow y=Ax\mathrm{e}^{6x}$	Most are able to write $\int\frac{1}{v}\mathrm{d}v=\ln\lvert v\rvert+c$.

(b) **(i)** **[1]**	$\frac{\mathrm{d}x}{\mathrm{d}t}=kx(P-x)$	
(ii) **[5]**	$\frac{\mathrm{d}x}{\mathrm{d}t}=kx(P-x)$ $\int\frac{1}{x(P-x)}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $\frac{1}{P}\int\frac{1}{x}+\frac{1}{P-x}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $\int\frac{1}{x}+\frac{1}{P-x}\,\mathrm{d}x=P\int k\,\mathrm{d}t$ $\ln\lvert x\rvert-\ln\lvert P-x\rvert=kPt+c$ $\ln x-\ln\left(P-x\right)=kPt+c,$ since $0<x<P$ $\int\frac{1}{-x^2+Px}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $\int\frac{1}{\left(\frac{P}{2}\right)^2-\left(x-\frac{P}{2}\right)^2}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $\frac{1}{P}\ln\left\lvert\frac{\frac{P}{2}+\left(x-\frac{P}{2}\right)}{\frac{P}{2}-\left(x-\frac{P}{2}\right)}\right\rvert=kt+C$ $\ln\left\lvert\frac{x}{P-x}\right\rvert=kPt+PC$ $\ln\left(\frac{x}{P-x}\right)=kPt+Pc$ $\frac{x}{P-x}=A\mathrm{e}^{kPt}$, where $A=\mathrm{e}^{Pc}$ $\ln\left(\frac{x}{P-x}\right)=kPt+c$ $\frac{x}{P-x}=A\mathrm{e}^{kPt}$, where $A=\mathrm{e}^{c}$ When $t=0, x=x_0$. So, $A=\frac{x_0}{P-x_0}$. $\frac{x}{P-x}=A\mathrm{e}^{kPt}$ $x=A\mathrm{e}^{kPt}\left(P-x\right)$ $x+xA\mathrm{e}^{kPt}=AP\mathrm{e}^{kPt}$ $x=\frac{AP\mathrm{e}^{kPt}}{1+A\mathrm{e}^{kPt}}$ $=\frac{\left(\frac{x_0}{P-x_0}\right)P\mathrm{e}^{kPt}}{1+\left(\frac{x_0}{P-x_0}\right)\mathrm{e}^{kPt}}$ $=\frac{Px_0\mathrm{e}^{kPt}}{\left(P-x_0\right)+x_0\mathrm{e}^{kPt}}$ $=\frac{Px_0}{\left(P-x_0\right)\mathrm{e}^{-kPt}+x_0}$	There is a significant number who are not able to solve this differential equation. A similar question was done in • Chap 8C notes Eg 11 • Tut 8C Qn 9, 11 •
(iii) **[1]**	As $t\to\infty, x\to P$. The whole population will be infected eventually.	As this is a contextual question, you will need to answer in words.

PROBABILITY AND STATISTICS (40 marks)

8 A group of hikers is to be chosen to represent three hiking teams in an annual hiking challenge. The group is to consist of 7 hikers and is chosen at random from a set of 10 hikers consisting of 2 hikers from team *A*, 3 hikers from team *B* and 5 hikers from team *C*. Find the number of ways in which the group can be chosen if

(i) there are no restrictions, [1]

(ii) there are at least 4 hikers from team *C*. [2]

(iii) there are at least 1 hiker from each team. [3]

<table>
<tr><td>(i)
[1]</td><td>Total selection $= {}^{10}C_7$
$= 120$</td><td></td></tr>
<tr><td>(ii)
[2]</td><td>Number of selections
$= {}^5C_4 \times {}^5C_3 + {}^5C_5 \times {}^5C_2$
[4 hikers from Team C & 3 hikers from others + 5 hikers from Team C & 2 hikers from others]
$= 60$</td><td>There are a few who wrote the answer as ${}^5C_4 \times {}^6C_3 = 100$. This is not correct, as there will many repeated counts. Refer to Tut S1A Q6.</td></tr>
<tr><td>(iii)
[3]</td><td>Number of selections
$= 120 - {}^8C_7 - {}^7C_7$
[No restriction − No hiker from team A − No hiker from team B]
$= 111$

Alternatively,
Number of selections
$= {}^2C_1 \times {}^8C_6 + {}^8C_5 - 1$
[choose 1 out of 2 from team A & 6 hikers from others
+ choose both hikers from team A & 5 hikers from others
− both hikers from team A are chosen & 5 hikers from team C]
$= 111$

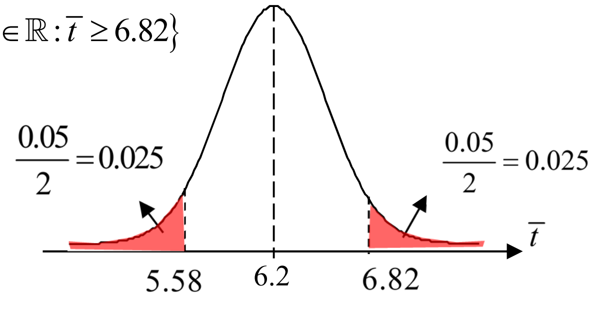
Alternatively,
<table>
<tr><td colspan="3">Number of hikers from Team</td><td rowspan="2">Number of ways</td></tr>
<tr><td><i>A</i></td><td><i>B</i></td><td><i>C</i></td></tr>
<tr><td>1</td><td>1</td><td>5</td><td>${}^2C_1 \times {}^3C_1 \times {}^5C_5 = 6$</td></tr>
<tr><td>1</td><td>2</td><td>4</td><td>${}^2C_1 \times {}^3C_2 \times {}^5C_4 = 30$</td></tr>
<tr><td>1</td><td>3</td><td>3</td><td>${}^2C_1 \times {}^3C_3 \times {}^5C_3 = 20$</td></tr>
<tr><td>2</td><td>1</td><td>4</td><td>${}^2C_2 \times {}^3C_1 \times {}^5C_4 = 15$</td></tr>
<tr><td>2</td><td>2</td><td>3</td><td>${}^2C_2 \times {}^3C_2 \times {}^5C_3 = 30$</td></tr>
<tr><td>2</td><td>3</td><td>2</td><td>${}^2C_2 \times {}^3C_3 \times {}^5C_2 = 10$</td></tr>
</table>
Total number of ways $= 6 + 30 + 20 + 15 + 30 + 10 = 111$</td><td></td></tr>
</table>

9 A study found that students at the National University of Singapore and the Nanyang Technological University sleep an average of 6.2 hours a day. Andy surveys 50 students from university *S* to estimate each person's daily sleeping hours. The sample mean is $\overline{t}$ hours and the sample standard deviation is 2.2 hours. A test is to be carried out at the 5% level of significance to draw conclusions on the average daily sleeping hours of students from university *S*.

Source: https://www.straitstimes.com/singapore/lack-of-sleep-is-nothing-to-yawn-about

(i) Find the set of values of $\overline{t}$ for which he will conclude that the students from university *S* sleep more than 6.2 hours on average. State the appropriate hypotheses for the test, defining any symbols you use. [5]

(ii) State the set of values of $\overline{t}$ for which he will conclude that the average sleeping hours of students from university *S* is not 6.2 hours. [1]

(i) **[5]**	Let T denote the number of sleeping hours of a randomly selected student from university S and let μ hours be the population mean sleeping time of the students from university *S*. Null hypothesis $H_0 : \mu = 6.2$ vs Alternative hypothesis $H_1 : \mu > 6.2$ An unbiased estimate for the population variance, $s^2 = \frac{50}{50-1}\left(2.2^2\right) = 4.93877551$ hours^2 Under H_0, $\overline{T} \sim N\left(6.2, \frac{4.93877551}{50}\right)$ approximately by Central Limit Theorem, since $n = 50$ is large. To conclude that the mean sleeping hours is more than 6.2 hours, i.e. for H_0 to be rejected, p-value $= P\left(\overline{T} \geq \overline{t}\right) \leq 0.05$. By G.C., $\overline{t} \geq 6.71695$ Therefore, the set of values of $\overline{t}$ is $\left\{\overline{t} \in \mathbb{R} : \overline{t} \geq 6.72\right\}$ (3 s.f.).	Many did not define μ properly, although this is done before in • Chap S5 Eg 4 • Tut S5 Qn 9 The given standard deviation in the question is the **sample** standard deviation. Many did not find the **unbiased estimate** for the population variance.
(ii) **[1]**	$H_0 : \mu = 6.2$ VS $H_1 : \mu \neq 6.2$ To conclude that the mean sleeping hours is not 6.2 hours, i.e. for H_0 to be rejected, $\overline{t}$ must lie in the critical region. i.e. $P\left(\overline{T} \leq \overline{t}\right) \leq \frac{0.05}{2}$ OR $P\left(\overline{T} \geq \overline{t}\right) \leq \frac{0.05}{2}$ $\overline{t} \leq 5.5840$ or $\overline{t} \geq 6.8160$ (5s.f.) Therefore, the required set of values of $\overline{t}$ is $\left\{\overline{t} \in \mathbb{R} : \overline{t} \leq 5.58\right\} \cup \left\{\overline{t} \in \mathbb{R} : \overline{t} \geq 6.82\right\}$ 	

10 For events A and B, it is given that $\mathrm{P}(A') = 0.3$ and $\mathrm{P}(B) = 0.55$.

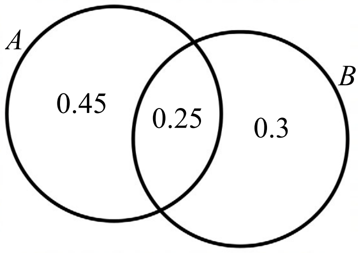
(i) Given that the events A and B are independent, find $\mathrm{P}(A \cap B)$. [1]

(ii) Given that events A and B are not independent, find the range of $\mathrm{P}(A \cup B)$. [2]

(iii) Given that $\mathrm{P}(A' \mid B) = 0.4$, find

(a) $\mathrm{P}(A \cup B)$, [3]

(b) $\mathrm{P}(A' \cup B')$. [2]

(i) **[1]**	$\mathrm{P}(A \cap B) = 0.7(0.55) = 0.385$, since A and B are independent	
(ii) **[2]**	$\mathrm{P}(A) = 0.7,\ \mathrm{P}(B) = 0.55$. The least value of $\mathrm{P}(A \cup B)$ occurs when $B \subseteq A$. Then the least value of $\mathrm{P}(A \cup B)$ is 0.7. The largest value of $\mathrm{P}(A \cup B)$ occurs when $\mathrm{P}(A \cup B) = 1$. This is possible by considering $\mathrm{P}(A \cap B) = 0.25$. (see Venn diagram below)  From **(i)**, when A and B are independent, $\mathrm{P}(A \cap B) = 0.385$. Then, the corresponding $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ $= 0.7 + 0.55 - 0.385$ $= 0865$ Since A and B are **not** independent, then we need to **exclude** $\mathrm{P}(A \cup B) = 0.865$. Thus, range of $\mathrm{P}(A \cup B)$ is $[0.7, 1] \setminus \{0.865\}$.	
(iii) **(a)** **[3]**	<u>**Method 1**</u> $\mathrm{P}(A' \mid B) = 0.4 \ \Rightarrow\ \mathrm{P}(A' \cap B) = 0.4 \times 0.55 = 0.22$ Thus, $\mathrm{P}(A \cup B) = \mathrm{P}(A' \cap B) + \mathrm{P}(A) = 0.22 + 0.7 = 0.92$ <u>**Method 2**</u> $\mathrm{P}(A \mid B) = 1 - P(A' \mid B) = 1 - 0.4 = 0.6$ $\mathrm{P}(A \cap B) = \mathrm{P}(B)\,\mathrm{P}(A \mid B) = (0.55)(0.6) = 0.33$ $\mathrm{P}(A \cup B) = P(A) + P(B) - P(A \cap B)$ $= 0.7 + 0.55 - 0.33$ $= 0.92$	
(iii) **(b)** **[2]**	$P(A' \cup B') = 1 - \mathrm{P}(A \cap B)$ $= 1 - 0.33$ $= 0.67$	Quite a significant number of students wrote: • $P(A' \cup B') = 1 - \mathrm{P}(A \cup B)$ This is not correct, although $P(A') = 1 - \mathrm{P}(A)$. See from the Venn diagram • $P(A' \cup B') = 1 - \mathrm{P}(A' \cap B')$

11 A policeman inspects each passenger in the car that he stops during a roadblock. The number of passengers in a randomly stopped car is the random variable X. The probability distribution of X is shown in the table below.

x	1	2	3	4	5
$\text{P}(X = x)$	0.32	0.43	0.16	0.07	k

(i) Find the value of k and determine $\text{E}(X)$. [2]

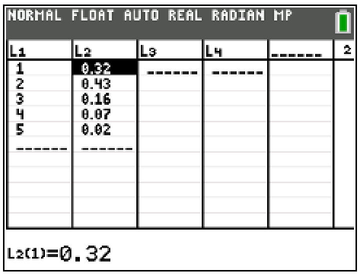
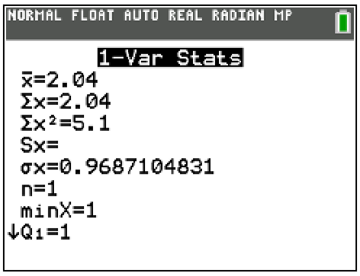
The policeman randomly stops 10 cars to inspect the passengers at a particular roadblock, and the total number of passengers inspected is the random variable Y.

(ii) Find $\text{E}(Y)$ and $\text{Var}(Y)$. [2]

The policeman also checks the car boot if the number of passengers in the car is not more than 2.

(iii) Find the probability that he checks more than 5 car boots at that roadblock. [2]

(iv) Find the probability that the last car is the 5th car boot that he checks at that roadblock. [2]

(i) **[2]**	$k = 1 - 0.32 - 0.43 - 0.16 - 0.07 = 0.02$ $\text{E}(X) = \sum x\,\text{P}(X = x) = 2.04$	
(ii) **[2]**	$Y = X_1 + X_2 + ... + X_{10}$ $\text{E}(Y) = 10\text{E}(X) = 20.4$  NORMAL FLOAT AUTO REAL RADIAN MP L1: 1, 2, 3, 4, 5 L2: 0.32, 0.43, 0.16, 0.07, 0.02 L2(1)=0.32  NORMAL FLOAT AUTO REAL RADIAN MP 1-Var Stats x̄=2.04 Σx=2.04 Σx²=5.1 Sx= σx=0.9687104831 n=1 minX=1 ↓Q1=1 $\text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2 = 0.96871^2 = 0.9384$ $\text{Var}(Y) = 10\text{Var}(X) = 9.384$	A few wrote • $\text{Var}(X) = \text{E}(X^2)$ • $\text{Var}(Y) = 10^2\text{Var}(X)$ It is stated on the cover page that “Give non-exact numerical answers correct to 3 significant figures”. So, $\text{Var}(Y) = 9.384$ is an **exact** answer, there is no need to correct to 3 s.f.
(iii) **[2]**	Let C be the number of car boots checked by the police, out of 10 cars. Then, $C \sim B(10, 0.32 + 0.43 = 0.75)$ $\text{P}(C > 5) = 1 - \text{P}(C \le 5) = 0.922$ (3.s.f.)	Common mistake • $\text{P}(C > 5) = 1 - \text{P}(C \le 4)$
(iv) **[2]**	Let D be the number of car boots checked by the police, out of 9 cars. Then, $D \sim B(9, 0.75)$ The required probability $= \text{P}(D = 4) \times 0.75$ $= 0.0389328 \times 0.75$ $= 0.0292$ (3 s.f.)	

12 In this question you should state the parameters of any distributions that you use.

A petrol station (Station A) finds that its daily sales, in litres, are normally distributed with mean 5400 and standard deviation 240.

(i) Sketch the distribution for the daily sales of petrol at Station A between 4500 litres and 6300 litres. [2]

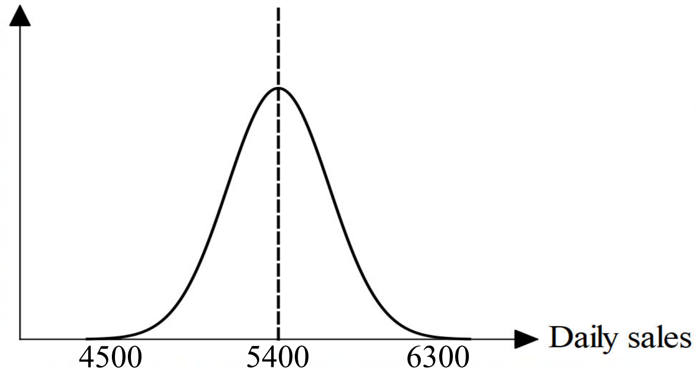
(ii) Find the expected number of days of the year (365 days) the daily sales of petrol at Station A exceed 5950 litres. [2]

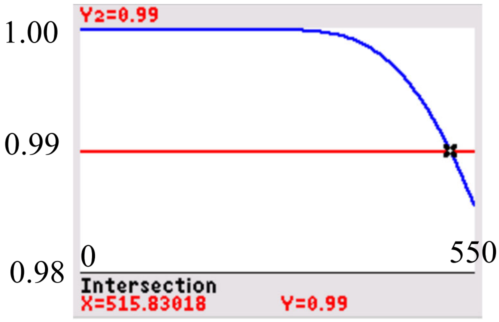
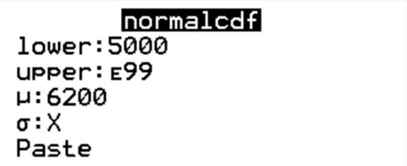
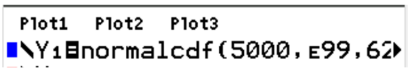
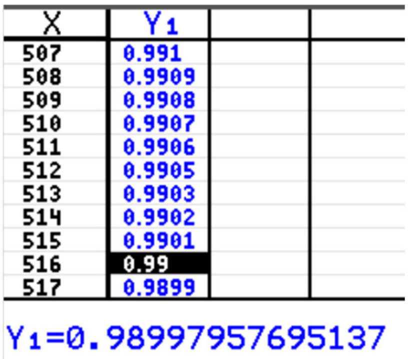
(iii) Find the probability that the total daily sales of petrol at Station A on 6 randomly chosen days is less than 32000 litres. [2]

The daily sales at another petrol station (Station B) are normally distributed with mean 6200 litres and standard deviation s litres.

(iv) The probability that daily sales of petrol at Station B on a randomly chosen day are more than 5000 litres is 0.99. Find the value of s. [3]

(v) Find the probability that, on a randomly chosen day, the daily sales of petrol on at Station B are more than half of that at Station A by at least 2000 litres. State an assumption needed for your calculation. [3]

(i) **[2]**		Most are able to draw the bell-shaped curve symmetrically about 5400. The probability should be approximately zero for more than 3 standard deviations from the mean – in this case below 4680 and above 6120. Thus, at 4500 and 6300, the curve should be very very close to the horizontal axis with the curve roughly horizontal at these points. See Chap S3 notes page 4. Similar question has appeared in recent A-level examination.
(ii) **[2]**	Let X be the daily sales of petrol at station A. Then $X \sim \mathrm{N}(5400, 240^2)$ The expected number of days is $365\,\mathrm{P}(X > 5950) = 4.001 \approx 4.00$ (3 s.f.)	
(iii) **[2]**	$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \sim \mathrm{N}(6 \times 5400, 6 \times 240^2)$ i.e. $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \sim \mathrm{N}\left(32400, \left(\sqrt{345600}\right)^2\right)$ $\mathrm{P}\left(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 < 32000\right) = 0.248$ (3 s.f.	Common mistake: some used 345600 as the standard deviation when keying into the GC, instead of $\sqrt{345600}$

(iv) **[3]**	**Method 1** Let Y be the daily sales of petrol at station B. Then $Y \sim \mathrm{N}(6200, s^2)$ $\mathrm{P}(Y>5000)=0.99$ $\mathrm{P}\left(Z>\frac{5000-6200}{s}\right)=0.99$ $\mathrm{P}\left(Z>-\frac{1200}{s}\right)=0.99$ From GC, we know that $\mathrm{P}(Z>-2.32635)=0.99$ Thus, $-\frac{1200}{s}=-2.32635$ $s=515.83$ (5 s.f.) $=516$ (3 s.f.) **Method 2** We can use GC to draw the graph of $y=\text{normalcdf(500,E99,6200,}x\text{)}$ and $y=0.99$, to see where they intersect. The x-coordinate of the point of intersection would represent the value of s.  From GC, $s=515.83$ (5 s.f.) $=516$ (3 s.f.) **Note**: this method is **not** encouraged because there is not much working. If you do not get your answer correctly, you may not get any marks at all as no method is seen.	Some found the value of s by using a table of values. However, this method is **not** suitable for this question  normalcdf lower:5000 upper:E99 μ:6200 σ:X Paste  Plot1 Plot2 Plot3 \Y1=normalcdf(5000,E99,62  X Y1 507 0.991 508 0.9909 509 0.9908 510 0.9907 511 0.9906 512 0.9905 513 0.9903 514 0.9902 515 0.9901 516 0.99 517 0.9899 Y1=0.98997957695137 • Table of values method should be used when the answer is an **integer**. What if the answer is s = 2.13 (3 s.f.)? • From the table above, it seems like s = 516 gives the probability to be exactly 0.99. However, when you move the cursor to the value, it is actually 0.98997… Because the space in the table can only show up to 5 digits, when rounded off to 4 d.p., it shows 0.99.
(v) **[3]**	$Y-\frac{1}{2}X \sim \mathrm{N}\left(6200-\frac{1}{2}\times 5400, 515.83^2+\frac{1}{4}\times 240^2\right)$ i.e. $Y-\frac{1}{2}X \sim \mathrm{N}\left(3500, \left(\sqrt{280480.6}\right)^2\right)$ $\mathrm{P}\left(Y-\frac{1}{2}X \geq 2000\right)=0.998$ (3 s.f.) Assumption: the daily sales of petrol at Station A and at Station B are independent of each other.	Some used the value of s to be 516, and this gives an over-estimated value for the variance of $Y-\frac{1}{2}X$. Some gave • $\mathrm{Var}\left(Y-\frac{1}{2}X\right)=515.83^2-\frac{1}{4}\times 240^2$ Quite a number of scripts missed the last part of the question, which is to state the assumption