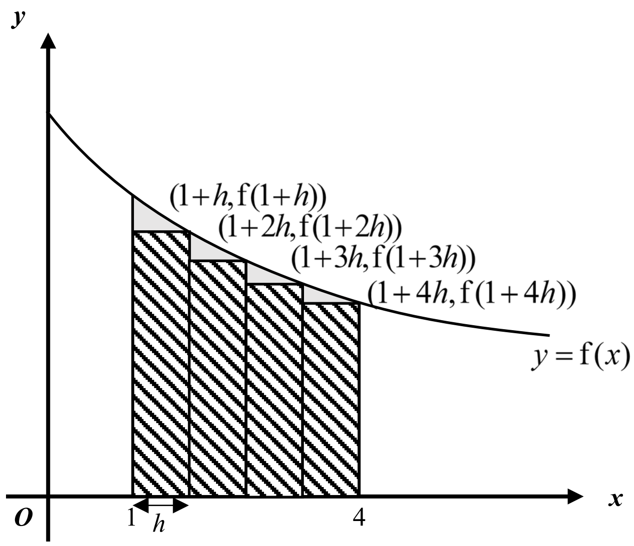
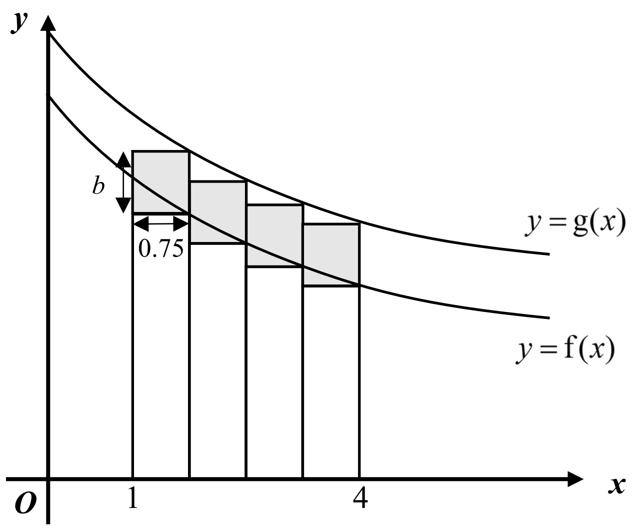
2023 JC2 H2MA Preliminary Examination P2 Solutions

	Solution
1a	$h=\dfrac{4-1}{4}=0.75$  From the diagram, Total area of shaded rectangles < Area under the curve from $x=1$ to $x=4$ $\therefore \sum_{n=1}^{4}\left[\mathrm{f}(1+nh)\right]h < \text{Area of } A$
b	$\sum_{n=0}^{3}\left[\mathrm{f}(1+nh)\right]h$ or $\sum_{n=1}^{4}\left[\mathrm{f}(0.25+nh)\right]h$ or $\sum_{n=1}^{4}\left[\mathrm{f}(4-nh)\right]h$ or $\sum_{n=1}^{4}\left[\mathrm{f}(1+(n-1)h)\right]h$
c	By GC, Lower Bound $=\sum_{n=1}^{4}\left[0.75\mathrm{f}(1+0.75n)\right]$ $=\sum_{n=1}^{4}\left[0.75(-2\ln(1+0.75n+1)+7)\right]$ $=13.0$

	Upper Bound $=\sum_{n=0}^{3}\left[0.75\mathrm{f}(1+0.75n)\right]$ $=\sum_{n=0}^{3}\left[0.75(-2\ln(1+0.75n+1)+7)\right]$ $=14.4$
d	(For students' understanding only)  Since translation by b units in the positive y-direction will result in an increase in the area by 4 rectangles (represented by c) with each length 0.75 units and breadth b units, $\therefore c=4\times 0.75b$ $c=3b$ <u>Alternatively,</u> $\sum_{n=1}^{4}\left[\mathrm{g}(1+nh)\right]h=\sum_{n=1}^{4}\left[\mathrm{f}(1+nh)+b\right]h$ $=\sum_{n=1}^{4}\left[\mathrm{f}(1+nh)\right]h+\sum_{n=1}^{4}bh$ $=\sum_{n=1}^{4}\left[\mathrm{f}(1+nh)\right]h+4bh$ $\therefore c=4b(0.75)$ $c=3b$
2a	**Method 1** $\left\|3\mathrm{i}-\sqrt{3}\right\|=\sqrt{3+9}=2\sqrt{3}$ $\arg\left(3\mathrm{i}-\sqrt{3}\right)=\pi-\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)=\frac{2\pi}{3}$ $\left\|1-\mathrm{i}\right\|=\sqrt{1+1}=\sqrt{2}$ $\arg\left(1-\mathrm{i}\right)=-\tan^{-1}\left(\frac{1}{1}\right)=-\frac{\pi}{4}$

	$\lvert z\rvert = \frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$ $\arg(z) = \frac{2\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{11\pi}{12}$ **Method 2 (NOT recommended)** $z = \frac{3\mathrm{i} - \sqrt{3}}{1-\mathrm{i}}$ $= \frac{\left(3\mathrm{i} - \sqrt{3}\right)(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}$ $= \frac{\left(-\sqrt{3}-3\right) + \left(3-\sqrt{3}\right)\mathrm{i}}{2}$ $\lvert z\rvert = \sqrt{\left[\frac{\left(-\sqrt{3}-3\right)}{2}\right]^2 + \left[\frac{\left(3-\sqrt{3}\right)}{2}\right]^2}$ $= \sqrt{\frac{3+9+6\sqrt{3}}{4} + \frac{3+9-6\sqrt{3}}{4}}$ $= \sqrt{6}$ $\arg(z) = \pi - \tan^{-1}\left(\frac{\left(3-\sqrt{3}\right)}{\left(\sqrt{3}+3\right)}\right) = \frac{11\pi}{12}$
b	$z = \frac{3\mathrm{i} - \sqrt{3}}{k-\mathrm{i}}$ $= \frac{\left(3\mathrm{i} - \sqrt{3}\right)(k+\mathrm{i})}{(k-\mathrm{i})(k+\mathrm{i})}$ $= \frac{\left(-\sqrt{3}k-3\right) + \left(3k-\sqrt{3}\right)\mathrm{i}}{k^2+1}$ $z = z^* \Rightarrow \operatorname{Im}(z) = 0$ $\frac{3k-\sqrt{3}}{k^2+1} = 0$ $k = \frac{\sqrt{3}}{3}$

3a

$$\ln\left(1-\frac{1}{r^2}\right)=\ln\left(\frac{r^2-1}{r^2}\right)$$
$$=\ln\left(\frac{(r-1)(r+1)}{r^2}\right)$$
$$=\ln(r-1)+\ln(r+1)-\ln r^2$$
$$=\ln(r-1)-2\ln r+\ln(r+1) \quad \text{(shown)}$$

b

$$S_n=\sum_{r=2}^{n}\ln\left(1-\frac{1}{r^2}\right)$$
$$=\sum_{r=2}^{n}\left[\ln(r-1)-2\ln r+\ln(r+1)\right]$$
$$\begin{aligned}
=\ &\ln 1-2\ln 2+\ln 3\\
&+\ln 2-2\ln 3+\ln 4\\
&+\ln 3-2\ln 4+\ln 5\\
&+\ln 4-2\ln 5+\ln 6\\
&+\quad \vdots\\
&+\ln(n-3)-2\ln(n-2)+\ln(n-1)\\
&+\ln(n-2)-2\ln(n-1)+\ln n\\
&+\ln(n-1)-2\ln n+\ln(n+1)
\end{aligned}$$
$$=-\ln 2-\ln n+\ln(n+1)$$
$$=\ln\left(\frac{n+1}{2n}\right)$$

c

$$S_n=\ln\left(\frac{n+1}{2n}\right)=\ln\left(\frac{1}{2}+\frac{1}{2n}\right)$$

As $n\to\infty$, $\frac{1}{2n}\to 0$. Hence, $S_\infty=\lim_{n\to\infty}\ln\left(\frac{1}{2}+\frac{1}{2n}\right)=\ln\frac{1}{2}$.

Considering $\left|S_n-S_\infty\right|\le 0.05$, using the GC,

n	$\left\|S_n-S_\infty\right\|$
19	0.0513 (>0.05)
20	0.0488 (≤ 0.05)
21	0.0465 (≤ 0.05)

Smallest $n = 20$.

4ai

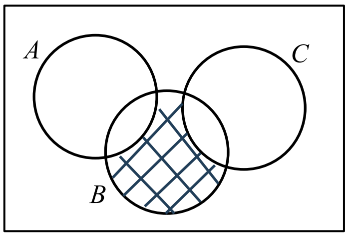
$$\underset{\sim}{v}\times\underset{\sim}{u}=\underset{\sim}{0}$$

- Either $\underset{\sim}{v}=\underset{\sim}{0}$
- Or $\underset{\sim}{v}\ne\underset{\sim}{0}$ (and given $\underset{\sim}{u}\ne\underset{\sim}{0}$), $\underset{\sim}{v}$ and $\underset{\sim}{u}$ are parallel (accept $\underset{\sim}{v}=k\underset{\sim}{u}$, where $k\ne 0$)

Combining both cases, $\underset{\sim}{v}=k\underset{\sim}{u}$, where $k\in\mathbb{R}$

aii	$\underset{\sim}{v}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}$, $\underset{\sim}{n}=\frac{1}{\sqrt{1+4+4}}\begin{pmatrix}1\\-2\\-2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ Alternative answer : $\underset{\sim}{n}=-\frac{1}{3}\begin{pmatrix}1\\-2\\-2\end{pmatrix}$
bi	$\overrightarrow{AB}=\underset{\sim}{b}-\underset{\sim}{a}$ $\overrightarrow{AC}=\left(4\underset{\sim}{a}-\frac{2}{3}\underset{\sim}{b}\right)-\underset{\sim}{a}=3\underset{\sim}{a}-\frac{2}{3}\underset{\sim}{b}$ Area of $\Delta ABC=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=14$ $\frac{1}{2}\left\|(\underset{\sim}{b}-\underset{\sim}{a})\times\left(3\underset{\sim}{a}-\frac{2}{3}\underset{\sim}{b}\right)\right\|=14$ $\left\|3\underset{\sim}{b}\times\underset{\sim}{a}-\frac{2}{3}\underset{\sim}{b}\times\underset{\sim}{b}-3\underset{\sim}{a}\times\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{a}\times\underset{\sim}{b}\right\|=28$ Since $\underset{\sim}{a}\times\underset{\sim}{a}=\underset{\sim}{0}$, $\underset{\sim}{b}\times\underset{\sim}{b}=\underset{\sim}{0}$ and $\underset{\sim}{b}\times\underset{\sim}{a}=-\underset{\sim}{a}\times\underset{\sim}{b}$, $\left\|-\frac{7}{3}\underset{\sim}{a}\times\underset{\sim}{b}\right\|=28$ $\frac{7}{3}\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|=28$ $\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|=12$ (Shown)
bii	Since $\|\mathbf{a}\|=5$, $\|\mathbf{b}\|=3$ $\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|=12 \quad\Rightarrow\quad \|\underset{\sim}{a}\|\|\underset{\sim}{b}\|\sin\angle AOB=12$ $15\sin\angle AOB=12$ $\sin\angle AOB=\frac{4}{5}$ $\angle AOB$ is obtuse $\quad\Rightarrow\quad \cos\angle AOB<0$ $\cos\angle AOB=-\sqrt{1-\left(\frac{4}{5}\right)^2}$ $=-\frac{3}{5}$

5a	$\dfrac{AB}{\sin\frac{\pi}{6}}=\dfrac{1}{\sin\left(\frac{5\pi}{6}-x\right)}=\dfrac{1}{\sin\frac{5\pi}{6}\cos x-\cos\frac{5\pi}{6}\sin x}$ $\dfrac{AB}{\frac{1}{2}}=\dfrac{1}{\frac{1}{2}\cos x-\left(-\frac{\sqrt{3}}{2}\right)\sin x}$ $\dfrac{AB}{\frac{1}{2}}=\dfrac{1}{\frac{1}{2}\left(\cos x+\sqrt{3}\sin x\right)}$ $AB=\dfrac{1}{\cos x+\sqrt{3}\sin x}$ (Shown)
b	$AB\approx\dfrac{1}{\left(1-\frac{1}{2}x^2\right)+\sqrt{3}x}$ $\approx\left[1+\left(\sqrt{3}x-\frac{1}{2}x^2\right)\right]^{-1}$ $\approx1-\left(\sqrt{3}x-\frac{1}{2}x^2\right)+\left(\sqrt{3}x-\frac{1}{2}x^2\right)^2$ $\approx1-\sqrt{3}x+\frac{1}{2}x^2+3x^2$ $\approx1-\sqrt{3}x+\frac{7}{2}x^2$
c	$\ln(px+q)=\ln\left[q\left(1+\frac{p}{q}x\right)\right]$ $=\ln q+\ln\left(1+\frac{p}{q}x\right)$ $=\ln q+\frac{p}{q}x+\cdots$ Comparing, $\ln q=1$ and $\frac{p}{q}=-\sqrt{3}$ $q=\mathrm{e}$ and $p=-\sqrt{3}\mathrm{e}$
6a	$\mathrm{P}(A\cap B)=1-\mathrm{P}(A'\cup B')$ $=1-0.82$ $=0.18$ $\mathrm{P}(A\mid B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ $0.4=\dfrac{0.18}{\mathrm{P}(B)}$ $\mathrm{P}(B)=0.45$

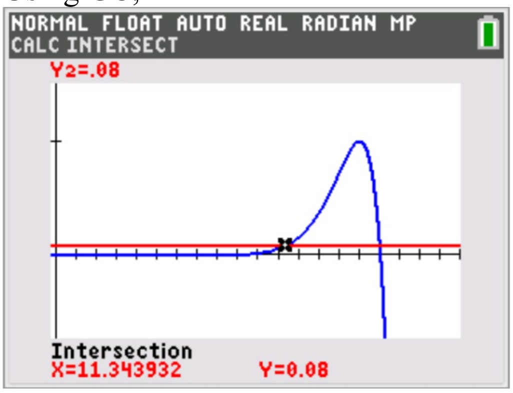
b	$\mathrm{P}(B')=1-\mathrm{P}(B)$ $=1-0.45$ $=0.55$ $A\cap B'\subseteq B'$ $\mathrm{P}(A\cap B')\le 0.55$
c	B & C independent. Hence $\mathrm{P}(B\cap C)=\mathrm{P}(B)\mathrm{P}(C)$ $=0.45\times 0.1$ $=0.045$ A & C are mutually exclusive.  Hence $\mathrm{P}(A'\cap B\cap C')$ $=\mathrm{P}(B)-\mathrm{P}(A\cap B)-\mathrm{P}(B\cap C)$ $=0.45-0.18-0.045$ $=0.225$
7a	Group the 2 red discs and the blue disc as one unit. Together with the remaining 6 green discs, there are 7 units. Required no. of arrangements $=\dfrac{7!}{6!}\times\dfrac{3!}{2!}$ $=21$
bi	$\mathrm{P}(R=0)=\dfrac{{}^2C_0\times{}^7C_3}{{}^9C_3}=\dfrac{5}{12}$ or $\dfrac{7}{9}\times\dfrac{6}{8}\times\dfrac{5}{7}=\dfrac{5}{12}$ $\mathrm{P}(R=1)=\dfrac{{}^2C_1\times{}^7C_2}{{}^9C_3}=\dfrac{1}{2}$ or $\dfrac{2}{9}\times\dfrac{7}{8}\times\dfrac{6}{7}\times 3=\dfrac{1}{2}$ $\mathrm{P}(R=2)=\dfrac{{}^2C_2\times{}^7C_1}{{}^9C_3}=\dfrac{1}{12}$ or $\dfrac{2}{9}\times\dfrac{1}{8}\times\dfrac{7}{7}\times 3=\dfrac{1}{12}$ or $1-\dfrac{5}{12}-\dfrac{1}{2}=\dfrac{1}{12}$ Probability distribution of R:

r	0	1	2
$\mathrm{P}(R=r)$	$\dfrac{5}{12}$	$\dfrac{1}{2}$	$\dfrac{1}{12}$

bii

r	0	1	2
$\mathrm{P}(R=r)$	$\dfrac{5}{12}$	$\dfrac{1}{2}$	$\dfrac{1}{12}$
Change in points	$0(9)+3(-3)$ $=-9$	$1(9)+2(-3)$ $=3$	$2(9)+1(-3)$ $=15$

	Expected change in points $=(-9)\mathrm{P}(R=0)+(3)\mathrm{P}(R=1)+(15)\mathrm{P}(R=2)$ $=(-9)\left(\frac{5}{12}\right)+(3)\left(\frac{1}{2}\right)+(15)\left(\frac{1}{12}\right)$ $=-1$ **<u>Alternative solution</u>** $\mathrm{E}(R)=0\left(\frac{5}{12}\right)+1\left(\frac{1}{2}\right)+2\left(\frac{1}{12}\right)=\frac{2}{3}$ Expected change in points $=9\times\mathrm{E}(R)-3\times\mathrm{E}(3-R)$ $=(9)\frac{2}{3}+(-3)\left(3-\frac{2}{3}\right)$ $=-1$
8a	Unbiased estimate of population mean, $\bar{x}=\frac{198.5}{35}+500$ ≈ 505.671 ≈ 506 (3 s.f.) Unbiased estimate of population variance, $s^2=\frac{1}{35-1}\left[7188-\frac{(198.5)^2}{35}\right]$ ≈ 178.3006 ≈ 178 (3 s.f.)
b	$\mathrm{H_o}: \mu=500$ $\mathrm{H_1}: \mu\neq 500$ Test at 2% significance level. Under $\mathrm{H_o}$,, since the sample size n = 35 is large, by Central Limit Theorem, $\bar{X}\sim \mathrm{N}\left(500, \frac{178.3006}{35}\right)$ approximately. Using GC, p-value = 0.011986 < 0.02 We reject $\mathrm{H_o}$ and conclude that there is sufficient evidence, at the 2% level of significance, that the mean mass of the packets of scallops is not 500 grams.
c	$s^2=\frac{40}{39}\left(11.7^2\right)$ $=140.4$ $\mathrm{H_0}: \mu=\mu_0$ (claim) $\mathrm{H_1}: \mu<\mu_0$

	Level of significance: 5% Under H_0, since the sample size n = 40 is large, by Central Limit Theorem, $\overline{X} \sim N\left(\mu_0, \dfrac{140.4}{40}\right)$ approximately. Since H_0 is not rejected, $\dfrac{510-\mu_0}{\sqrt{140.4/40}} > -1.64485$ $0 < \mu_0 < 513.08$ (2 d.p.)
9a	$\mathrm{P}(\text{Obtaining 'hit'}) = \dfrac{\text{Area of inner circle}}{\text{Area of outer circle}}$ $= \dfrac{\pi r^2}{\pi(15^2)}$ $= \dfrac{r^2}{225}$
bi	Let X be the random variable denoting the number of 'hits' out of 8 throws. Then $X \sim B\left(8, \dfrac{r^2}{225}\right)$ $\mathrm{P}(X > 6) \le 0.08$ $\mathrm{P}(X = 7) + \mathrm{P}(X = 8) \le 0.08$ ${}^8C_7\left(\dfrac{r^2}{225}\right)^7\left(1-\dfrac{r^2}{225}\right)+\left(\dfrac{r^2}{225}\right)^8 \le 0.08$ Using GC,  $r \le 11.3439$ (6 s.f.) Therefore, $r \le 11.3$ (3 s.f.)
bii	$\dfrac{6^2}{225} = 0.16$ Let Y be the random variable denoting the number of 'hits' out of 4 throws. Then $Y \sim B(4, 0.16)$

Required probability

$= \mathrm{P}(Y=1)\times(0.16)$

$= 0.060693$

$= 0.0607 \quad (3 \text{ s.f.})$

Alternative solution

Required probability

$= \left[(0.16)(0.84)^3 \times 4\right](0.16)$

= 0.0607 (3 s.f.)

biii Let W be the random variable denoting the number of 'hits' out of n throws. Then

$W \sim \mathrm{B}(n, 0.16)$

$\mathrm{P}(W \geq 1) \geq 0.7$

$1-\mathrm{P}(W=0) \geq 0.7$

$\mathrm{P}(W=0) \leq 0.3$

By GC,

n	$\mathrm{P}(W=0)$
6	$0.3513 > 0.3$
7	$0.2951 < 0.3$
8	$0.2479 < 0.3$

Therefore, the least value of n is 7.

Alternative solution

$\mathrm{P}(W \geq 1) \geq 0.7$

$1-\mathrm{P}(W=0) \geq 0.7$

$\mathrm{P}(W=0) \leq 0.3$

$\binom{n}{0}(0.16)^0(0.84)^n \leq 0.3$

$(0.84)^n \leq 0.3$

$n \geq \dfrac{\ln 0.3}{\ln 0.84}$

$n \geq 6.9054$

Therefore, the least value of n is 7.

biv $X \sim \mathrm{B}(8,\ 0.16)$

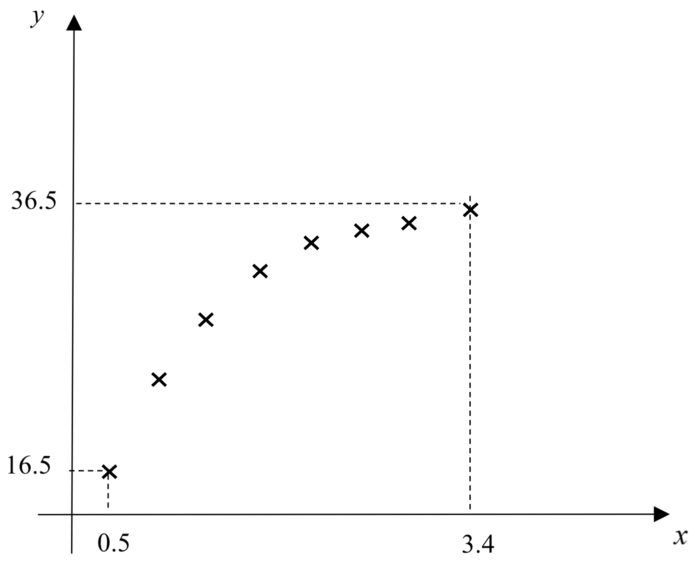
Required probability

$= \mathrm{P}\left(X < 2(8\times 0.16) \mid X > 8\times 0.16\right)$

$= \mathrm{P}\left(X < 2.56 \mid X > 1.28\right)$

	$=\dfrac{\mathrm{P}(1.28<X<2.56)}{\mathrm{P}(X>1.28)}$ $=\dfrac{\mathrm{P}(X=2)}{\mathrm{P}(X\geq 2)}$ $=\dfrac{0.25181}{1-0.62559}$ $=0.673$ (3 s.f.)
10a	Let X be the mass (kg) of a randomly chosen pumpkin and Y be the mass (kg) of a randomly chosen cabbage. $X\sim \mathrm{N}(3.7,0.4^2)$ $\quad Y\sim \mathrm{N}(0.8,0.12^2)$ $\mathrm{P}(3.2<X<m)=0.6$ $\mathrm{P}(X<m)-\mathrm{P}(X<3.2)=0.6$ $\mathrm{P}(X<m)=0.70565$ $m=3.9163$ $=3.916$ (3 d.p.) (Shown)
b	$\mathrm{P}(X>m)=1-0.70565=0.29435$ Let W be the number of pumpkins with a mass greater than m kg, out of 20 pumpkins. $W\sim \mathrm{B}(20,0.29435)$ $\mathrm{P}(W>5)=1-\mathrm{P}(W\leq 5)$ $=0.56178$ $=0.562$ (3 s.f.)
c	Let $\overline{X}=\dfrac{X_1+X_2+...+X_n}{n}$. $\overline{X}\sim \mathrm{N}\left(3.7,\dfrac{0.4^2}{n}\right)$ $\mathrm{P}(\overline{X}\leq 3.8)>0.95$ From GC, (see table below) Least value of n is 44. **<u>Alternative solution:</u>**

n	$\mathrm{P}(\overline{X}\leq 3.8)$
43	0.949 (< 0.95)
44	0.951 (> 0.95)
45	0.953 (> 0.95)

	$\mathrm{P}\left(\overline{X} \le 3.8\right) > 0.95$ $\mathrm{P}\left(Z \le \dfrac{3.8-3.7}{\sqrt{0.4^2/n}}\right) > 0.95$ From GC, $\dfrac{0.1\sqrt{n}}{0.4} \ge 1.64485$ $\sqrt{n} \ge 6.5794$ $n \ge 43.289$ Least value of n is 44.
d	Let S be the total selling price of three randomly chosen pumpkins and four randomly chosen cabbages. $S = 5\left(X_1 + X_2 + X_3\right) + 3\left(Y_1 + Y_2 + Y_3 + Y_4\right)$ $\mathrm{E}\left(S\right) = 5\left(3\times 3.7\right) + 3\left(4\times 0.8\right) = 65.1$ $\mathrm{Var}\left(S\right) = 5^2\left(3\times 0.4^2\right) + 3^2\left(4\times 0.12^2\right) = 12.5184$ $S \sim \mathrm{N}\left(65.1, 12.5184\right)$ $\mathrm{P}(S < 60) = 0.0747$ (3 s.f.)
11a	Value of b: For <u>every increment of 1 kg in the average mass of fried chicken consumed per week,</u> the <u>mass of a 30-year-old male</u> is expected to <u>increase by 0.765 kg.</u> Value of a: When the <u>average mass of fried chicken consumed per week is 0 kg,</u> the expected <u>mass of a 30-year-old male is 62.3 kg.</u>
b	
c	For $y = p + qx$, $r = 0.898$

	For $y = p + q\mathrm{e}^{-x}$, $r = -0.997$ Since $\lvert r \rvert = 0.997$ for $y = p + q\mathrm{e}^{-x}$ is **closer to 1** than $\lvert r \rvert = 0.898$ for $y = p + qx$, $y = p + q\mathrm{e}^{-x}$ is a better model. $y = 38.3 - 34.7\mathrm{e}^{-x}$
d	$y = 38.270 - 34.727\mathrm{e}^{-x}$ When $x = 2.3$, $y = 38.270 - 34.727\mathrm{e}^{-x}$ $= 38.270 - 34.727\mathrm{e}^{-2.3}$ $= 34.8$ (3 sf) The estimate is reliable as $\lvert r \rvert = 0.997$ **is close to 1** AND $x = 2.3$ is **within the given data range for *x*** OR the estimate is obtained by **interpolation** of the data.
ei	There will be <u>no change</u> in the product moment correlation coefficients since <u>y is multiplied by a positive constant</u>.
eii	$z = \frac{95}{100} y$ $\frac{100}{95} z = 38.270 - 34.727\mathrm{e}^{-x}$ $z = 36.4 - 33.0\mathrm{e}^{-x}$