H2 Mathematics (9758)
Chapter 6A 3D Vector Geometry (Lines)
Assignment Solutions

1 **N2008/I/11(modified)**

The cartesian equation for two lines are

$$\frac{x}{1}=\frac{y+2}{2}=\frac{z-5}{-1} \text{ and } \frac{x-1}{-1}=\frac{y+3}{-3}=\frac{z-4}{1}.$$

(i) Show that the lines intersect and state the point of intersection. [4]

(ii) Find the acute angle between the lines. [2]

Q1	Solutions
(i)	$\frac{x}{1}=\frac{y+2}{2}=\frac{z-5}{-1} \Rightarrow \mathbf{r}=\begin{pmatrix}0\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}, \lambda\in\mathbb{R}$ $\frac{x-1}{-1}=\frac{y+3}{-3}=\frac{z-4}{1} \Rightarrow \mathbf{r}=\begin{pmatrix}1\\-3\\4\end{pmatrix}+\mu\begin{pmatrix}-1\\-3\\1\end{pmatrix}, \mu\in\mathbb{R}$ At the point of intersection, $\begin{pmatrix}\lambda\\-2+2\lambda\\5-\lambda\end{pmatrix}=\begin{pmatrix}1-\mu\\-3-3\mu\\4+\mu\end{pmatrix}$ for some $\lambda,\mu\in\mathbb{R}$ $\lambda=1-\mu$ (1) $-2+2\lambda=-3-3\mu$ (2) $5-\lambda=4+\mu$ (3) Solving using GC, $\lambda=4, \mu=-3$ Since there is a consistent value of λ and μ satisfying the equations, the two lines intersect at the point. Point of intersection: $(4,6,1)$
(ii)	Let θ be the acute angle between the two lines. $\theta=\cos^{-1}\left\lvert\frac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}-1\\-3\\1\end{pmatrix}}{\sqrt{6}\sqrt{11}}\right\rvert=\cos^{-1}\frac{8}{\sqrt{6}\sqrt{11}}=10.0^\circ$ (1 d.p.)

2 2011/SRJC/II/2(modified)

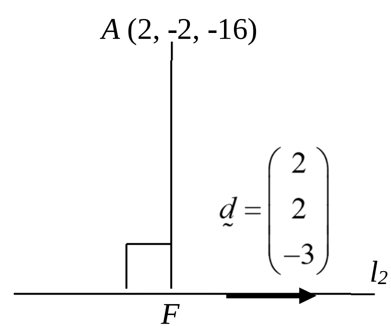
With respect to origin O, the position vector of the point A is $2\mathbf{i} - 2\mathbf{j} - 16\mathbf{k}$ and the line l_1 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}, \ \mu \in \mathbb{R}$.

(i) Find the exact length of projection of $\overrightarrow{OA}$ onto line l_1. [2]

(ii) Find the position vector of the foot of the perpendicular from the point A to the line l_1. [4]

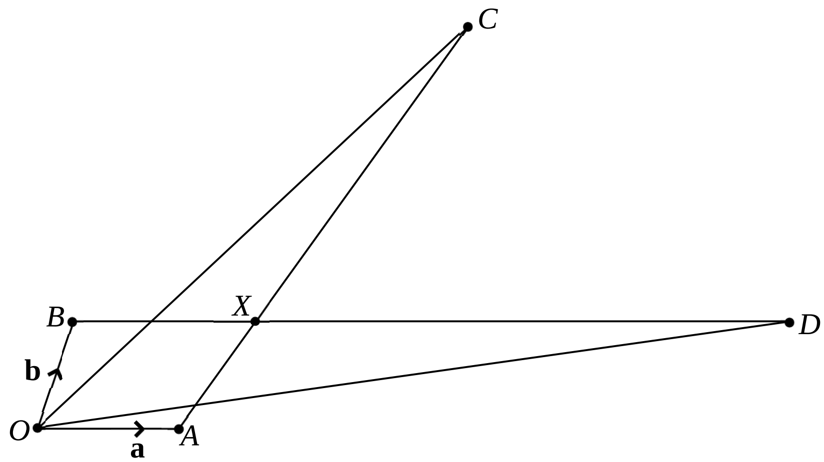
(iii) Find the perpendicular distance from the point A to the line l_1. [1]

(iv) Find the position vector of the point A', which is the reflection of point A in the line l_1. [2]

Q2	Solutions
(i)	Required length of projection $= \dfrac{\left\lvert \begin{pmatrix} 2 \\ -2 \\ -16 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \right\rvert}{\sqrt{2^2 + 2^2 + (-3)^2}}$ $= \dfrac{48}{\sqrt{17}}$
(ii)	Let F be the foot of the perpendicular from A to l_1. Since F lies on $l_1 \Rightarrow \overrightarrow{OF} = \begin{pmatrix} 1+2\mu \\ 2\mu \\ -4-3\mu \end{pmatrix}$, for some $\mu \in \mathbb{R}$ $\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} 1+2\mu \\ 2\mu \\ -4-3\mu \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -16 \end{pmatrix} = \begin{pmatrix} -1+2\mu \\ 2\mu+2 \\ 12-3\mu \end{pmatrix}$ $\overrightarrow{AF} \perp \underset{\sim}{d} \Rightarrow \overrightarrow{AF} \cdot \underset{\sim}{d} = 0$ $\begin{pmatrix} -1+2\mu \\ 2\mu+2 \\ 12-3\mu \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = 0$ $(-2+4-36) + \mu(4+4+9) = 0$ $\mu = 2$ $\Rightarrow \overrightarrow{OF} = \begin{pmatrix} 1+2(2) \\ 2(2) \\ -4-3(2) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -10 \end{pmatrix}$ 
(iii)	$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 4 \\ -10 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -16 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$ The perpendicular distance from A to l_1 $\left\lvert \overrightarrow{AF} \right\rvert = \sqrt{(3)^2 + (6)^2 + (6)^2} = \sqrt{81} = 9$ units

(iv)	F is the mid-point of AA' $\Rightarrow \overrightarrow{OF} = \dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ So, $\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA} = 2\begin{pmatrix} 5 \\ 4 \\ -10 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -16 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -4 \end{pmatrix}$

3 N2019/II/5



With reference to the origin *O*, the points *A*, *B*, *C* and *D* are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{a} + 4\mathbf{b}$ and $\overrightarrow{OD} = \mathbf{b} + 5\mathbf{a}$. The lines *BD* and *AC* cross at *X* (see diagram).

(i) Express $\overrightarrow{OX}$ in terms of **a** and **b**. [4]

The point *Y* lies on *CD* and is such that the points *O*, *X* and *Y* are collinear.

(ii) Express $\overrightarrow{OY}$ in terms of **a** and **b** and find the ratio $OX : OY$. [6]

Q3	Solutions
(i)	$\overrightarrow{BD} = \mathbf{b} + 5\mathbf{a} - (\mathbf{b}) = 5\mathbf{a}$ $\overrightarrow{AC} = 2\mathbf{a} + 4\mathbf{b} - \mathbf{a} = \mathbf{a} + 4\mathbf{b}$ Note the presentation of equation of lines: - Label it clearly so that you are able to identify it easily for future uses - $l_{BD} = \mathbf{b} + \lambda\mathbf{a}$ is wrong presentation. $l_{BD}: \mathbf{r} = \mathbf{b} + \lambda\mathbf{a},\ \lambda \in \mathbb{R}$ $l_{AC}: \mathbf{r} = \mathbf{a} + \mu(\mathbf{a} + 4\mathbf{b}),\ \mu \in \mathbb{R}$ Concept: Point of intersection between 2 intersecting lines Since the lines *BD* and *AC* cross at *X*, we equate the two lines: $\mathbf{b} + \lambda\mathbf{a} = \mathbf{a} + \mu(\mathbf{a} + 4\mathbf{b})$. Since **a** and **b** are non-zero and non-parallel vectors, $1 = 4\mu \Rightarrow \mu = \frac{1}{4}$ $\lambda = \mu + 1 \Rightarrow \lambda = \frac{5}{4}$ Learning Point: If **a** and **b** are non-zero and non-parallel vectors and $\lambda\,\mathbf{a} = \mu\,\mathbf{b}$ for some $\lambda, \mu \in \mathbb{R}$, then $\lambda = 0$ and $\mu = 0$. Substitute $\lambda = \frac{5}{4}$ into l_{BD} to be the position vector of *X* which is the point of intersection of the two lines. $\overrightarrow{OX} = \mathbf{b} + \frac{5}{4}\mathbf{a}$

(ii)

$\overrightarrow{CD} = \mathbf{b} + 5\mathbf{a} - (2\mathbf{a} + 4\mathbf{b})$

$= 3\mathbf{a} - 3\mathbf{b}$

$l_{CD} : \mathbf{r} = 2\mathbf{a} + 4\mathbf{b} + \alpha(\mathbf{a} - \mathbf{b}), \alpha \in \mathbb{R}$

Since Y lies on l_{CD}, $\overrightarrow{OY} = 2\mathbf{a} + 4\mathbf{b} + \alpha(\mathbf{a} - \mathbf{b})$ for some $\alpha \in \mathbb{R}$.

Given that points O, X and Y are collinear,

let $\overrightarrow{OY} = \beta\overrightarrow{OX}$

$= \beta\left[\mathbf{b} + \frac{5}{4}\mathbf{a}\right]$

$\therefore \beta\left[\mathbf{b} + \frac{5}{4}\mathbf{a}\right] = 2\mathbf{a} + 4\mathbf{b} + \alpha(\mathbf{a} - \mathbf{b})$

Learning Point:
Points A, B, and C are collinear
$\Leftrightarrow \overrightarrow{AB} = \lambda\overrightarrow{AC}$, for some $\lambda \in \mathbb{R}, \lambda \neq 0$.

Since **a** and **b** are non-zero and non-parallel vectors,

$\beta = 4 - \alpha \quad --(1)$

$\frac{5}{4}\beta = 2 + \alpha \quad --(2)$

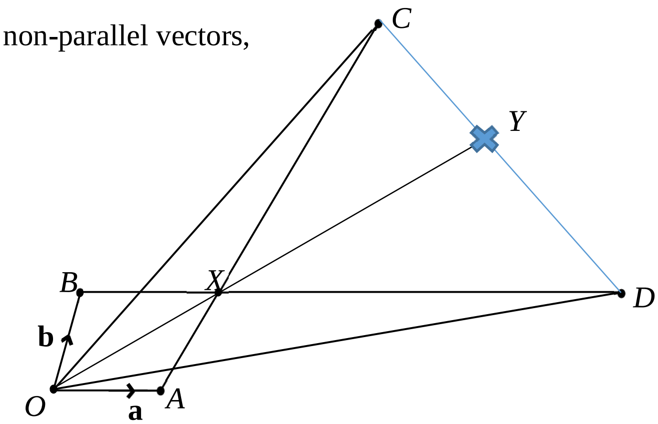
Using GC,

$\beta = \frac{8}{3}, \alpha = \frac{4}{3}$

$\overrightarrow{OY} = \frac{8}{3}\left[\mathbf{b} + \frac{5}{4}\mathbf{a}\right]$

$\overrightarrow{OY} = \frac{8}{3}\overrightarrow{OX}$

$\therefore OX : OY = 3 : 8$



Alternative:

Observe that point Y is the point of intersection between line CD and line OX.

$l_{CD} : \mathbf{r} = 2\mathbf{a} + 4\mathbf{b} + \alpha(\mathbf{a} - \mathbf{b}), \alpha \in \mathbb{R}$

$l_{OX} : \mathbf{r} = \beta\left(\mathbf{b} + \frac{5}{4}\mathbf{a}\right), \beta \in \mathbb{R}$

Since the lines CD and OX cross at Y, we equate the two lines:

$\therefore \beta\left[\mathbf{b} + \frac{5}{4}\mathbf{a}\right] = 2\mathbf{a} + 4\mathbf{b} + \alpha(\mathbf{a} - \mathbf{b})$

Since **a** and **b** are non-zero and non-parallel vectors,

$\beta = 4 - \alpha \quad --(1)$

$\frac{5}{4}\beta = 2 + \alpha \quad --(2)$

Using GC,

$\beta = \frac{8}{3}, \alpha = \frac{4}{3}$

$\overrightarrow{OY} = \frac{8}{3}\mathbf{b} + \frac{10}{3}\mathbf{a} = \frac{8}{3}\left[\mathbf{b} + \frac{5}{4}\mathbf{a}\right]$

$= \frac{8}{3}\overrightarrow{OX}$

$\therefore OX : OY = 3 : 8$

Take note of presentation:
There is no division of vectors, hence writing

$\frac{\overrightarrow{OX}}{\overrightarrow{OY}} = \frac{\mathbf{b} + \frac{5}{4}\mathbf{a}}{\frac{8}{3}\mathbf{b} + \frac{10}{3}\mathbf{a}}$ is **WRONG**

$\frac{\overrightarrow{OX}}{\overrightarrow{OY}} = \frac{3}{8}$