YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
Higher 2

CANDIDATE NAME

CG

MATHEMATICS

9758/01

Paper 1

30 AUGUST 2023

Candidates answer on the Question Paper. **3 hours**
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your CG, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiners' Use

Question	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Presentation	
Total	/ 100

1 A function is defined as $\mathrm{f}(x)=\ln\left(\dfrac{a}{x+3a}\right)$, where $a>1$. Describe fully a sequence of transformations which transforms the curve $y=\ln x$ onto the curve $y=\mathrm{f}(x)$. [4]

2 **Do not use a calculator in answering this question.**

One of the roots of the equation $z^3-3z^2+kz+13=0$, where k is real, is $2+3\mathrm{i}$.

(a) Find the other roots of the equation and the value of k. [4]

(b) Deduce the roots of the equation $-\mathrm{i}w^3+3w^2+k\mathrm{i}w+13=0$. [2]

3 **(a)** Find the exact roots of the equation $\left|3x^2+8x-3\right|=3-x$. [3]

(b) On the same axes, sketch the curves with equations $y=\left|3x^2+8x-3\right|$ and $y=3-x$.

Hence solve exactly the inequality $\left|3x^2+8x-3\right|\geq 3-x$. [4]

4 A curve C has parametric equations

$$x=\sqrt{3}\sin 2t,\quad y=4\cos^2 t,\quad \text{for } 0\leq t\leq \pi.$$

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=k\sqrt{3}\tan 2t$, where k is a constant to be determined. [2]

(b) Find the equations of the tangents to C at the points where $t=\dfrac{\pi}{4}$ and $t=\dfrac{\pi}{3}$. [4]

(c) Find the acute angle between these two tangents. [2]

5 **(a)** An infinite geometric progression has first term a and common ratio r, where a and r are non-zero. The sum of all the terms after the nth term of the progression is equal to twice the nth term. Show that the sum to infinity of the progression is three times the first term. [3]

(b) The positive integers, starting at 1, are grouped into sets, as follows.

$$\{1\},\{2,3\},\{4,5,6\},\ldots,$$

where there are r integers in the rth set.

(i) Find, in terms of r, the first integer and the last integer in the rth set. [3]

(ii) Prove that the sum of the integers in the rth set is $\dfrac{1}{2}r\left(1+r^2\right)$. [2]

6 The region R is bounded by the two curves with equations $y=1+\cos 2x$ and $y=1+\cos^2 x$, where $0\leq x\leq\pi$.

(a) Find the exact area of region R. [4]

(b) Show that $\cos^4 x=\dfrac{1}{8}\left(3+4\cos 2x+\cos 4x\right)$. [2]

(c) Find the exact volume of the solid formed by rotating R through 2π radians about the x-axis. [4]

7 **(a)** **(i)** It is given that $\frac{\mathrm{d}y}{\mathrm{d}x}=(4x-y+2)^2$. Using the substitution $v=4x-y$, show that the differential equation can be transformed to $\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{f}(v)$, where the function $\mathrm{f}(v)$ is to be found. [2]

(ii) Hence, given that $y=-2$ when $x=0$, solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=(4x-y+2)^2$, to find y in terms of x. [4]

(b) The variables x and y are connected by the following differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^{-2x}+\sqrt{x}.$$

(i) Find the general solution, giving your answer in the form $y=\mathrm{f}(x)$. [2]

(ii) Find the solution curve where the tangent at the origin is parallel to the line $y=2x+3$. [2]

8 The function f is defined by

$$\mathrm{f}:x\mapsto\frac{1}{x^2-4ax+5a^2},\qquad x\in\mathbb{R},\ x<k,$$

where a and k are positive constants.

(a) Find the largest possible value of k, in terms of a, such that f^{-1} exists. [2]

It is now given that $k=a$ and $a>1$.

(b) Show that the composite function f^2 exists. [2]

(c) Find $\mathrm{f}^{-1}(x)$ in terms of x and a. [3]

(d) On the same diagram, sketch the graphs of f and f^{-1}, giving the equations of any asymptotes. [4]

9 **(a)** **(i)** Find $\int x^2\sqrt{x^3+1}\,\mathrm{d}x$. [2]

(ii) Hence, find the exact value of $\int_{-1}^{0} x^5\sqrt{x^3+1}\,\mathrm{d}x$. [3]

(b) Use the substitution $u=1+\mathrm{e}^x$ to find $\int \mathrm{e}^{2x}\sqrt{1+\mathrm{e}^x}\,\mathrm{d}x$. [4]

(c) Find $\int(2+\tan 5x)\cos 5x\sin 3x\,\mathrm{d}x$. [3]

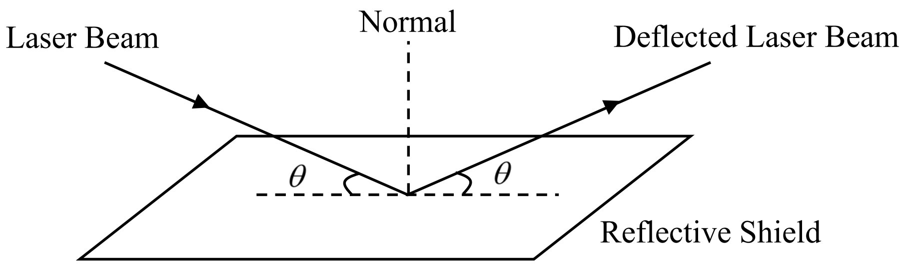
10 A security system in a museum uses laser to protect a valuable artefact. The path of the laser beam can be represented by the line passing through the points $(0,-3,1)$ and $(2,-1,2)$. The museum also installed a reflective shield, represented by the plane passing through the points $(6,9,3)$, $(-2,13,1)$ and $(4,10,0)$.

(a) Show that a cartesian equation of the plane representing the reflective shield is $x+2y=24$. [2]

(b) Find θ, the acute angle between the laser beam and the reflective shield. [2]

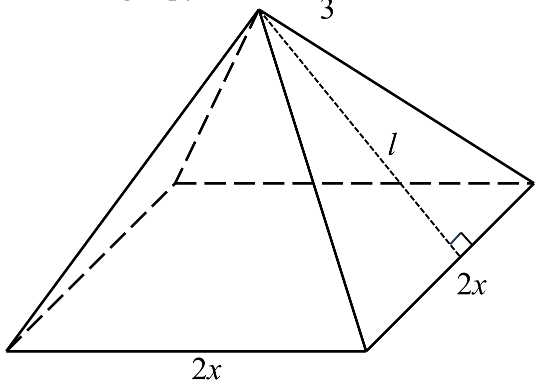
(c) Find the position vector of the point where the laser beam hits the reflective shield. [2]

The reflective shield deflects the laser beam such that the angle between the laser beam and the reflective shield equals to the angle between the deflected laser beam and the reflective shield. The laser beam, the deflected laser beam and the normal of the reflective shield lie in the same plane (see diagram).



(d) Find a vector equation of the line representing the deflected laser beam. [6]

11 [It is given that the volume of a right pyramid is $\frac{1}{3}\times$ base area $\times$ height.]



A manufacturer designs a decorative item in the shape of a right pyramid with five parts. The square base has sides $2x$ cm. The four triangular faces each has a base length of $2x$ cm and height of l cm. The five parts are joined together as shown in the diagram. The item is made of material of negligible thickness.

The manufacturer determines that the total external surface area of the item must be 144 cm^2 and that the total volume of the item, $V\text{ cm}^3$, should be as large as possible.

(a) Show that $V = 8\sqrt{36x^2 - 2x^4}$. [3]

(b) Use differentiation to show that the maximum value of V is $72\sqrt{2}\text{ cm}^3$, obtained when the value of x is 3 cm. Prove that the value of V is a maximum. [4]

The item is now manufactured with the value of x being 3 cm. To make the item glow in the dark, the item is to be filled entirely with fluorescent liquid. The liquid is injected into the item at a rate of 10 cm^3 per second.

(c) Find the rate of increase of the depth of the liquid after 6 seconds. [5]