2015 VJC JC2 Prelim Paper 2 Solutions

Q1(i) Given $e^y = \frac{1}{2} + \sin(2x) - - - (1)$ Differentiating with respect to x, $e^y \frac{dy}{dx} = 2\cos(2x) - - - (2)$ Differentiating again with respect to x, $e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 = -4\sin(2x)$ $e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 = -4\left(e^y - \frac{1}{2}\right)$ $e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 + 4e^y = 2$ $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 4 = 2e^{-y} - - - (3)$ **Q1(ii)** Differentiating result in (i) with respect to x, $\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) = -2e^{-y}\frac{dy}{dx} - - - (4)$ At $x = 0$, (1): $e^y = \frac{1}{2} \Rightarrow y = \ln\left(\frac{1}{2}\right)$ (2): $\frac{1}{2}\frac{dy}{dx} = 2\cos(0) \Rightarrow \frac{dy}{dx} = 4$ (3): $\frac{d^2y}{dx^2} + (4)^2 + 4 = \frac{2}{\frac{1}{2}} \Rightarrow \frac{d^2y}{dx^2} = -16$ (4): $\frac{d^3y}{dx^3} + 2(4)(-16) = -2(2)(4) \Rightarrow \frac{d^3y}{dx^3} = 112$ $\therefore y = \ln\frac{1}{2} + 4x + \left(\frac{-16}{2!}\right)x^2 + \left(\frac{112}{3!}\right)x^3 + ...$ $= \ln\frac{1}{2} + 4x - 8x^2 + \frac{56}{3}x^3 + ...$ **Q1(iii)** $y = \ln\left[\frac{1}{2} + \sin(2x)\right]$ $= \ln\left[\frac{1}{2} + (2x) - \frac{(2x)^3}{3!} + ...\right]$ $= \ln\left[\frac{1}{2} + 2x - \frac{4x^3}{3} + ...\right]$ $= \ln\frac{1}{2} + \ln\left[1 + \left(4x - \frac{8x^3}{3}\right) + ...\right]$	

$= \ln\frac{1}{2} + \left(4x - \frac{8x^3}{3}\right) - \frac{1}{2}\left(4x - \frac{8x^3}{3}\right)^2 + \frac{1}{3}\left(4x - \frac{8x^3}{3}\right)^3 + \ldots$ $= \ln\frac{1}{2} + 4x - \frac{8x^3}{3} - \frac{1}{2}\left(16x^2 + \ldots\right) + \frac{1}{3}\left[\left(4x\right)^3 + \ldots\right] + \ldots$ $= \ln\frac{1}{2} + 4x - 8x^2 + \frac{56}{3}x^3 + \ldots$ **<u>Alternative</u>** $y = \ln\left[\frac{1}{2} + \sin\left(2x\right)\right]$ $= \ln\frac{1}{2} + \ln\left[1 + 2\sin 2x\right]$ $= \ln\frac{1}{2} + \left(2\sin 2x\right) - \frac{1}{2}\left(2\sin 2x\right)^2 + \frac{1}{3}\left(2\sin 2x\right)^3 + \ldots$ $= \ln\frac{1}{2} + 2\left[2x - \frac{\left(2x\right)^3}{3!} + \ldots\right] - \frac{4}{2}\left[2x - \frac{\left(2x\right)^3}{3!} + \ldots\right]^2$ $+ \frac{8}{3}\left[2x - \frac{\left(2x\right)^3}{3!} + \ldots\right]^3 + \ldots$ $= \ln\frac{1}{2} + 4x - \frac{8x^3}{3} - 2\left(4x^2 + \ldots\right) + \frac{8}{3}\left(8x^3 + \ldots\right) + \ldots$ $= \ln\frac{1}{2} + 4x - 8x^2 + \frac{56}{3}x^3 + \ldots$	
Q2) $\overrightarrow{OA} = \mathbf{a},\ \overrightarrow{OB} = \mathbf{a} + \mathbf{c},\ \overrightarrow{OC} = \mathbf{c}$ **Q2(i)** $\overrightarrow{ON} = \frac{\overrightarrow{OB} + \overrightarrow{OX}}{2}$ $\overrightarrow{ON} = \frac{1}{2}\left[\mathbf{a} + \mathbf{c} + 4\mathbf{c} - 3\mathbf{a}\right]$ $= \frac{1}{2}\left[5\mathbf{c} - 2\mathbf{a}\right]$ $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ $= \overrightarrow{OB} + 4\overrightarrow{AC}$ $= \mathbf{a} + 4\left(\mathbf{c} - \mathbf{a}\right)$ $= 4\mathbf{c} - 3\mathbf{a}$ **Q2(ii)** Area of triangle *OAB* = $\frac{1}{2}\left\|\overrightarrow{OA} \times \overrightarrow{OB}\right\|$ $2 = \frac{1}{2}\left\|\mathbf{a} \times \left(\mathbf{a} + \mathbf{c}\right)\right\|$ $= \frac{1}{2}\left\|\mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{c}\right\|$ $= \frac{1}{2}\left\|\mathbf{a} \times \mathbf{c}\right\| \quad \left(\because \mathbf{a} \times \mathbf{a} = \underset{\sim}{0}\right)$ $\Rightarrow \left\|\mathbf{a} \times \mathbf{c}\right\| = 4$	

Area of triangle $AXB = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AX}\right|$

$$= \frac{1}{2}\left|\mathbf{c}\times 4(\mathbf{c}-\mathbf{a})\right|$$
$$= 2\left|\mathbf{c}\times\mathbf{c}-\mathbf{c}\times\mathbf{a}\right|$$
$$= 2\left|\mathbf{a}\times\mathbf{c}\right|$$
$$= 2(4)$$
$$= 8$$

Q2(iii) $\left|\overrightarrow{AX}\times\frac{\overrightarrow{BX}}{\left|\overrightarrow{BX}\right|}\right|$ is the length of perpendicular from A to BX.

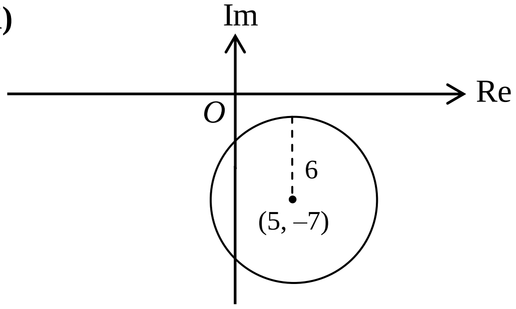
Area of triangle $AXB = 8$

$$\frac{1}{2}\left|\overrightarrow{AX}\times\frac{\overrightarrow{BX}}{\left|\overrightarrow{BX}\right|}\right|\left|\overrightarrow{BX}\right| = 8$$

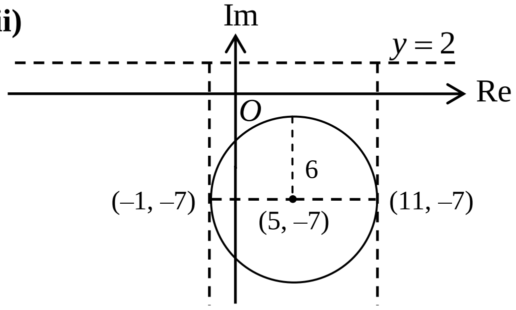
$$\left|\overrightarrow{AX}\times\frac{\overrightarrow{BX}}{\left|\overrightarrow{BX}\right|}\right| = \frac{16}{\left|\overrightarrow{BX}\right|}$$
$$= \frac{16}{\left|\overrightarrow{OX}-\overrightarrow{OB}\right|}$$
$$= \frac{16}{\left|4\mathbf{c}-3\mathbf{a}-\mathbf{a}-\mathbf{c}\right|}$$
$$= \frac{16}{\left|3\mathbf{c}-4\mathbf{a}\right|}$$

$\therefore k = 16,\ m = 3,\ n = 4$

Q3(i)

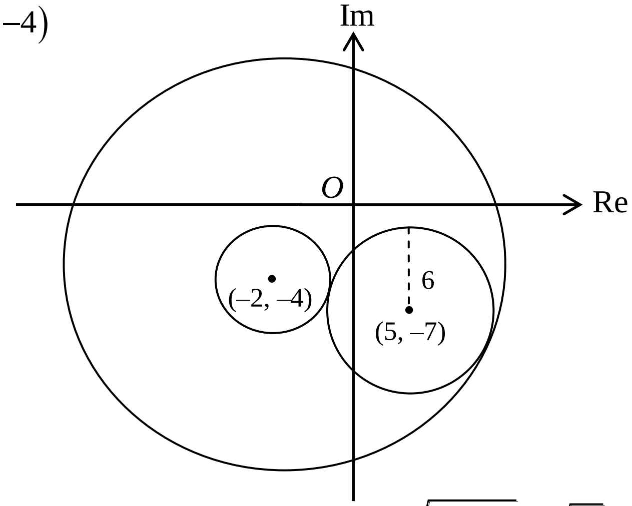


Q3(ii)



$\{a \in \mathbb{R} : -1 < a < 11\}$

Q3(iii) $|z+2+4\text{i}|=k$ is a circle with radius k and centre $(-2,-4)$



Distance between the two centres $=\sqrt{7^2+3^2}=\sqrt{58}$

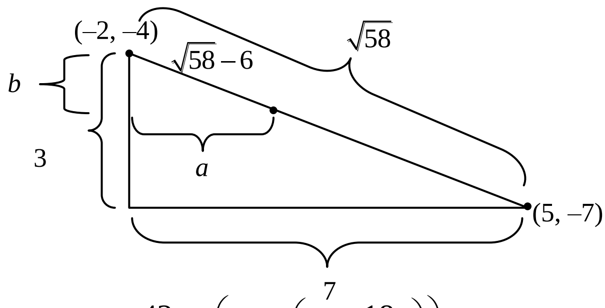
Possible exact values of k

$=\sqrt{58}-6$ and $\sqrt{58}+6$

For $k=\sqrt{58}-6$

$\frac{a}{7}=\frac{\sqrt{58}-6}{\sqrt{58}} \Rightarrow a=7-\frac{42}{\sqrt{58}}$

$\frac{b}{3}=\frac{\sqrt{58}-6}{\sqrt{58}} \Rightarrow b=3-\frac{18}{\sqrt{58}}$



$\therefore z=-2+7-\frac{42}{\sqrt{58}}+\left(-4-\left(3-\frac{18}{\sqrt{58}}\right)\right)\text{i}$

$=\left(5-\frac{42}{\sqrt{58}}\right)+\left(\frac{18}{\sqrt{58}}-7\right)\text{i}$

Similarly for $k=\sqrt{58}-6$, $z=\left(5+\frac{42}{\sqrt{58}}\right)-\left(\frac{18}{\sqrt{58}}+7\right)\text{i}$

Q4(i) $\frac{dy}{dx}=\frac{x^2\left(\frac{1}{x}\right)-(\ln x)(2x)}{x^4}=\frac{x-2x\ln x}{x^4}$ Let $\frac{dy}{dx}=0 \quad \Rightarrow x-2x\ln x=0$ $\therefore x=0$ or $\ln x=\frac{1}{2}$ $(\text{NA} \because x>0)$ $\Rightarrow x=\sqrt{e}$ When $x=\sqrt{e}$, $y=\frac{1/2}{(\sqrt{e})^2}=\frac{1}{2e}$ $\therefore$ Coordinates of A is $\left(\sqrt{e},\frac{1}{2e}\right)$ **Q4(ii)** Required area $=\int_1^2 y\,dx$ $=\int_1^2 \frac{\ln x}{x^2}\,dx$ $=\left[-\frac{\ln x}{x}\right]_1^2-\int_1^2\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$ $=-\frac{1}{2}\ln 2+\left[-\frac{1}{x}\right]_1^2$ $=\frac{1}{2}-\frac{1}{2}\ln 2$ **Q4(iii)** Required volume $=\pi\left(\frac{1}{2e}\right)^2\left(\sqrt{e}-1\right)-\pi\int_1^{\sqrt{e}} y^2\,dx$ $=\pi\left(\frac{1}{2e}\right)^2\left(\sqrt{e}-1\right)-\pi\int_1^{\sqrt{e}}\left(\frac{\ln x}{x^2}\right)^2 dx$ $=0.0245$ **Q4(iv)** $r=\frac{2e\ln x}{x^2}$. From (i) and the curve, $x>1 \Rightarrow 0<\frac{\ln x}{x^2}\le\frac{1}{2e}$ $\therefore x>1 \Rightarrow 0<\frac{2e\ln x}{x^2}\le 1$ Since $\lvert r\rvert\le 1$, S_∞ does not exist. So I disagree with him.	

Statistics

Q5(i) A stratified random sample of 60 students can be obtained by sampling 2 groups from the 2 levels as shown:	

Level	Number to be sampled
Year One	$\frac{540}{540+660}\times 60 = 27$
Year Two	$\frac{660}{540+660}\times 60 = 33$

Simple random sampling is used for the selection of students in each stratum. To select the 27 students from the Year One group, we first randomly assign each Year One student a distinct number from 1 to 540. Then generate 27 numbers using a random number generator and select the students corresponding to the 27 numbers generated. Repeat the procedure for the selection of the Year Two group.

Q5(ii) A better sample could be achieved by increasing the number of strata, example 4 strata instead of 2 : Female JC1, Male JC1, Female JC2 and Male JC2.

Q6(i) number of ways that 6 couples are seated together in a circle
$= 5!\times 2^6 = 7680$

Q6(ib) no. of ways that 12 people are seated in three distinct rows of four seats without restriction $= 12!$
no. of ways that 12 people are seated in a circle with 0 female in one of the rows $= \left({}^3C_1\right)\left({}^6C_2\right)(8!)(4!)$

Hence required no. of ways $= 12! - \left({}^3C_1\right)\left({}^6C_2\right)(8!)(4!)$
$= 435456000$

Q7(i) $\mathrm{P}\left(A \text{ wins on her turn}\right) = \frac{{}^5C_3}{{}^{13}C_3} = \frac{5}{143}$

$\mathrm{P}\left(A \text{ didn't win on her turn}\right) = 1 - \frac{{}^5C_3}{{}^{13}C_3} = \frac{138}{143}$

$\mathrm{P}\left(B \text{ didn't win on her turn}\right) = 1 - \frac{{}^8C_3}{{}^{13}C_3} = \frac{115}{143}$

$$\mathrm{P}\left(A \text{ wins}\right) = \frac{5}{143} + \left(\frac{138}{143}\right)\left(\frac{115}{143}\right)\left(\frac{5}{143}\right) + \left[\left(\frac{138}{143}\right)\left(\frac{115}{143}\right)\right]^2\left(\frac{5}{143}\right) + \dots$$

$$= \frac{\frac{5}{143}}{1 - \left[\left(\frac{138}{143}\right)\left(\frac{115}{143}\right)\right]}$$

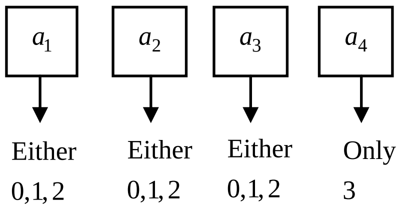
$$= \frac{715}{4579} = 0.156 \text{ (3 s f)}$$

Q7(ii) $\mathrm{P}\left(A \text{ wins on 1st draw} \mid A \text{ wins the game}\right)$

$$= \frac{\mathrm{P}\left(A \text{ wins on 1st draw}\right)}{\mathrm{P}\left(A \text{ wins the game}\right)}$$

$$= \frac{5/143}{715/4579}$$

$$= 0.224$$



No. of possible draw sequence $= 3\times3\times3\times1 = 27$

Q8(i) (1) Meteors occur at a constant average rate.
(2) Meteors occur independently.

Q8(ii) Let X be the number of meteors seen by Amy in 4 minutes

$$X \sim \mathrm{Po}(8)$$

$\mathrm{P}(X = 6) = 0.122$

Q8(iii) Let Y be the number of meteors seen by Amy in 2 minutes

$$Y \sim \mathrm{Po}(4)$$

Required probability $= \mathrm{P}(Y_1 = 3)\times\mathrm{P}(Y_2 = 3) = 0.0382$

Q8(iv) Since part (iii) is one of the possible solutions for part (ii), hence answer to (ii) is greater than answer to (iii).

Q8(v) Let W be the number of meteors seen by Amy in n minutes

$$W \sim \mathrm{Po}(2n)$$

Since $n > 10 \quad \Rightarrow 2n > 20$,

$$W \sim \mathrm{N}(2n, \quad 2n) \text{ approx}$$

$\mathrm{P}(W > 3n) < 0.005 \quad \Rightarrow \mathrm{P}(W > 3n + 0.5) < 0.005$

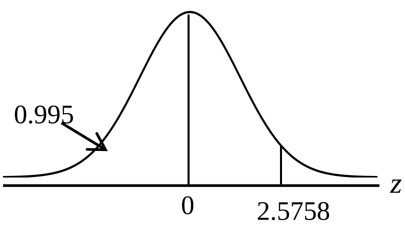
(continuity correction)

$$\Rightarrow \mathrm{P}\left(Z > \frac{3n + 0.5 - 2n}{\sqrt{2n}}\right) < 0.005$$

$$\Rightarrow \mathrm{P}\left(Z < \frac{n + 0.5}{\sqrt{2n}}\right) > 0.995$$

From the GC, $\mathrm{P}(Z < 2.3263) =$

$$\therefore \frac{n + 0.5}{\sqrt{2n}} > 2.57583$$



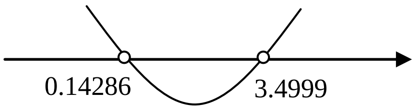
$n - 3.64277\sqrt{n} + 0.5 > 0$

Consider $n - 3.64277\sqrt{n} + 0.5 = 0$

$$\sqrt{n} = \frac{3.64277 \pm \sqrt{(3.64277)^2 - 4(1)(0.5)}}{2} = 0.14286 \text{ or } 3.4999$$

$(\sqrt{n} - 0.14286)(\sqrt{n} - 3.4999) > 0$

$\sqrt{n} < 0.14286$ or $\sqrt{n} > 3.4999$



$(\text{NA} \because n > 10)$

$\Rightarrow n > 12.249$

$\therefore$ smallest value of $n = 13$

Q9) Let $w = x - 160$, then $\sum w = 480$ and $\sum w^2 = 8837$

$$\bar{w} = \frac{480}{150} = 3.2 \text{ and } s_w^2 = \frac{1}{149}\left[8837 - \frac{(480)^2}{150}\right] = 49$$

$x = w + 160 \Rightarrow \bar{x} = \bar{w} + 100$ and $s_x^2 = s_w^2$

$\bar{x} = 3.2 + 160 = 163.2$, $s_x^2 = s_w^2 = 49$

Let μ cm be the population mean of X.

H_0 : $\mu = 162$

H_1 : $\mu > 162$

Level of sig : 5%

Test Statistic : When H_0 is true, $Z = \dfrac{\bar{X} - 162}{\left(\sqrt{\dfrac{49}{150}}\right)}$

Computation : $\bar{x} = 163.2$, $s^2 = 49$, p – value = 0.0179

Conclusion: Since p – value = 0.0179 $\leq$ 0.05, H_0 is rejected at 5% level of significance. Hence there is sufficient evidence that the mean height of 162 cm is an understated value.

At 5% significance level" means there is 0.05 probability that the test will conclude that the mean height of 162 cm is an understated value when actually it is not an understated value.

Q9(i) A t-test is carried out instead.

We need to assume that the height of a female student follows a normal distribution.

Q9(ii)

Yes. For an upper tail test, H_o is rejected if $\dfrac{m-162}{\frac{s}{\sqrt{n}}} \geq c$, where c is the critical value.

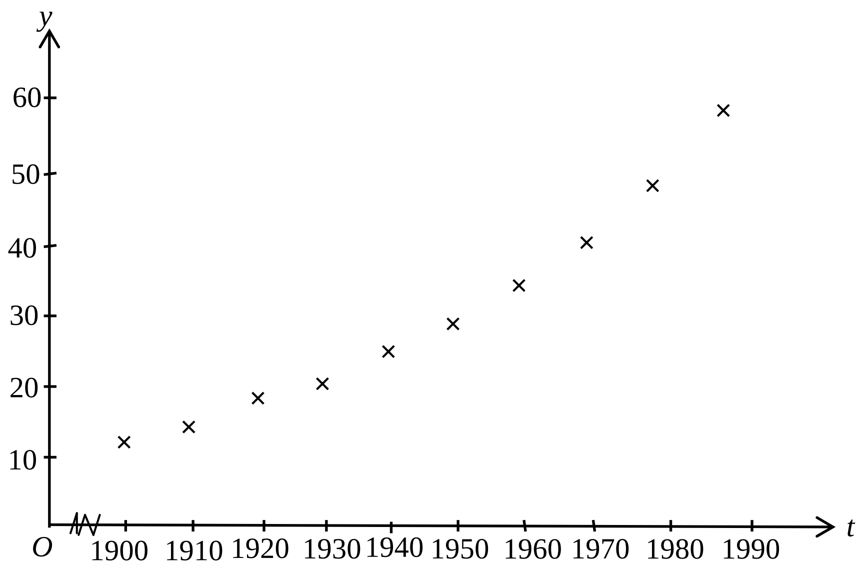
Since H_0 is not rejected, $m-162 < c\left(\dfrac{s}{\sqrt{n}}\right)$.

Given: $\bar{x}$ for 2^{nd} test is less than m, i.e. $\bar{x} < m$

$\Rightarrow \bar{x}-162 < m-162 < c\left(\dfrac{s}{\sqrt{n}}\right)$

Therefore H_0 will also not be rejected for 2^{nd} test (because both test have the same unbiased estimates for population variance and n).

Q10(i)



Q10(ii) The product moment correlation coefficient between t and y is $r = 0.97656$.

Points on the scatter diagram do not seem to lie close to a straight line with positive gradient. Therefore a value of 0.97656 (close to 1) does not necessarily mean that the best model for the relationship between t and y is $y = c + dt$.

Q10(iiia) $y = ae^{bt} \Rightarrow \ln y = \ln a + bt$

$Y = A + Bt \Rightarrow Y = \ln y,\ A = \ln a$ and $B = b$

Q10(iiib) The product moment correlation between t and Y is $r = 0.99946$.

Since $r = 0.99946$ is closer to $+1$ than $r = 0.97656$, thus $Y = A + Bt$ is the better model.

Q10(iiic) The estimated regression line of Y on t is $Y = -28.43362 + 0.01631435t$

i.e. $Y \approx -28.4 + 0.0163t$

For the year 2020, $Y = -28.43362 + 0.01631435(2020)$ $\therefore \ln y = 4.521367$ $\Rightarrow y = 91.961$ Estimated population in the year 2020 is 92.0 millions. The estimation is obtained by extrapolation since 2020 lies outside the data range of 1900 to 1990 and thus estimate is not reliable.	
Q11) Let G g and F g be the mass of a randomly chosen Granny Smith apple and Fuji apple respectively. $G \sim \mathrm{N}(150, 11.7^2) \qquad F \sim \mathrm{N}(180, 15.2^2)$ **Q11(i)** $\mathrm{E}(F_1 - F_2) = \mathrm{E}(F_1) - \mathrm{E}(F_2) = 0$ $\mathrm{Var}(F_1 - F_2) = 2\mathrm{Var}(F) = 462.08$ $F_1 - F_2 \sim \mathrm{N}(0, 462.08)$ $\mathrm{P}(\lvert F_1 - F_2 \rvert \geq 20)$ $= \mathrm{P}(F_1 - F_2 \leq -20) + \mathrm{P}(F_1 - F_2 \geq 20)$ $= 0.352$ **Q11(ii)** Let \$$T$ be the cost of 6 Granny Smith apples and 4 Fuji apples $T = \left(\frac{3.5}{1000}\right)(G_1 + \ldots + G_6) + \left(\frac{5}{1000}\right)(F_1 + \ldots + F_4)$ $\mathrm{E}(T) = \left(\frac{3.5}{1000}\right)(6)\mathrm{E}(G) + \left(\frac{5}{1000}\right)(4)\mathrm{E}(F) = 6.75$ $\mathrm{Var}(T) = \left(\frac{3.5}{1000}\right)^2(6)\mathrm{Var}(G) + \left(\frac{5}{1000}\right)^2(4)\mathrm{Var}(F)$ $= 0.033165$ $T \sim \mathrm{N}(6.75, 0.033165)$ $\mathrm{P}(T > 6.5) = 0.91509 \approx 0.915$ **Q11(iii)** We assume that the masses of ALL the apples are independent of one another. **Q11(iv)** $G_1 + \ldots + G_6 \sim \mathrm{N}(6 \times 150, 6 \times 11.7^2)$ $\mathrm{P}(G_1 + \ldots + G_6 < 950) = 0.959477$ Let X be the number of bags (out of 55) each with mass more than 950g. $X \sim \mathrm{B}(55, 1 - 0.959477)$ i.e. $X \sim \mathrm{B}(55, 0.040523)$ Since $n = 55$ is large and p is small such that $np = 2.22877 < 5$ $X \sim \mathrm{Po}(2.22877)$ approx	

$\mathrm{P}(55-X<48)=\mathrm{P}(X>7)$ $=1-\mathrm{P}(X\le 7)$ $=0.00214$	