RAFFLES INSTITUTION
2023 YEAR 6 PRELIMINARY EXAMINATION

FURTHER MATHEMATICS **9649/02**
Paper 2 **September 2023**
3 hours

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet and a graph paper booklet will be provided with this question paper. You should follow the instructions on the front cover of both booklets. If you need additional answer paper or graph paper ask the invigilator for a continuation booklet or graph paper booklet.

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

[Turn over

Section A: Pure Mathematics [50 marks]

1 **Do not use a calculator in answering this question.**

(i) Use de Moivre's theorem to prove that

$$\tan 6\theta = \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}.$$ [3]

(ii) Show that $\tan^2\left(\frac{\pi}{5}\right) = 5 - 2\sqrt{5}$. [4]

2 The matrices **A** and **B** are defined as follows.

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \text{ with } \theta \geq 0 \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The transformations T_1 and T_2 from $\mathbb{R}^2$ to $\mathbb{R}^2$ are defined by

$$\mathrm{T}_1 : \begin{pmatrix} x \\ y \end{pmatrix} \to \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathrm{T}_2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) By considering $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$ for $r > 0$ and $\alpha \in [0, 2\pi]$, describe the geometrical transformation T_1, explaining your answer. [2]

(b) Describe T_2 geometrically. [1]

Let matrix **M** be $\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$.

(c) Without using a calculator, state the smallest positive integer value of k such that $\mathbf{M}^k$ gives the identity matrix and explain your answer. [2]

(d) Given that $\mathbf{C} = \mathbf{M}\mathbf{B}\mathbf{M}^{-1}$, find the Cartesian equations of the two lines through the origin which are invariant under the transformation given by the matrix **C**. [3]

3 Pharmacokinetics is the study of the time course of absorption, distribution, metabolism, and excretion of a medication in the human body. When a medication is taken orally, it is dissolved and absorbed into the gastrointestinal (GI) tract. It is then diffused into the bloodstream through the distribution stage and eliminated from the bloodstream by the kidney and the liver through metabolism and excretion.

Let the number of units of medication present in the GI tract and the bloodstream at time t hours after an oral dose is administered be given by g and b respectively.

Using this model, it is given that $\frac{\mathrm{d}g}{\mathrm{d}t} = 1 - \beta g$ and $\frac{\mathrm{d}b}{\mathrm{d}t} = \beta g - \alpha b$, where α and β are positive constants representing the rate of elimination and distribution respectively, with $\alpha \neq \beta$.

(i) By setting up a second order differential equation for b, find the general solution for b in terms of α, β and t. [5]

In the case of the medication paracetamol, $\alpha = 0.2$ and $\beta = 0.7$. When $t = 0$, $b = 7.6$ and $\frac{\mathrm{d}b}{\mathrm{d}t} = -1.12$.

(ii) Find b in terms of t. [2]

(iii) For the effect of paracetamol in the body to be significant, the next dose needs to be administered when the amount of the medication in the bloodstream next falls below 5.69 units. How many hours later should the next dose be given for the model in part **(ii)**? [2]

[Turn over

4 Let α be a constant angle $0 < \alpha < \frac{\pi}{2}$. A curve C with the equation

$$r = \mathrm{e}^{\theta \cot \alpha}$$

in polar coordinates is called an *equiangular spiral*.

(i) Sketch the curve C, for $0 \leq \theta \leq \pi$. [2]

(ii) Now consider the part of C where $0 \leq \theta \leq \frac{\pi}{2}$.

Find, by differentiation, the value of θ at the point furthest from the half-line $\theta = \frac{\pi}{2}$, giving your answer in terms of α. You do not need to show that this value gives the maximum distance. [2]

(iii) Find the gradient of the tangent at the point $P(r, \theta)$. By using the answer to **(ii)** or otherwise, deduce the acute angle between the tangent at P and the line OP. [3]

(iv) Find an expression, in terms of α, for the length of the arc of the curve C, for $0 \leq \theta \leq \pi$, simplifying your answer. [3]

5 The function $y = \mathrm{f}(x)$ satisfies the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos^2 x - 2xy$ and $\mathrm{f}(0) = 0$.

The value of $\mathrm{f}(\alpha)$ is to be estimated, where α is a small positive number, using 2 methods.

(a) Use two steps of improved Euler method to determine an approximation to $\mathrm{f}(\alpha)$ in terms of α. [6]

(b) Now consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(1 - \frac{x^2}{2}\right)^2 - 2xy$, where $y = 0$ when $x = 0$.

(i) Given that

$$I_n = \int_0^{\alpha} x^n \mathrm{e}^{x^2} \, \mathrm{d}x,$$

where $n \geq 0$, show that, for integers $n \geq 2$,

$$I_n = \frac{1}{2}\alpha^{n-1}\mathrm{e}^{\alpha^2} - \frac{1}{2}(n-1)I_{n-2}.$$ [3]

(ii) Solve the differential equation and obtain, in terms of α, the value of y when $x = \alpha$. [5]

By substituting $\alpha = 0.1$, discuss the relative merits of the two methods employed to obtain these approximations. [2]

Section B: Probability and Statistics [50 marks]

6 In a particular school, the proportion of students who bring their own cutlery for their meals is p. Amy collects a large random sample of n students and calculates that a symmetric 95% confidence interval for p is $(0.229, 0.371)$. Bob collects a different, independent, random sample of 100 students and finds that 36 of them bring their own cutlery.

Based on all $n+100$ students in the two samples, it is desired to find a symmetric $\alpha\%$ confidence interval for p of width within 0.1. Find the largest possible value of α correct to 1 decimal place. [5]

7 The academic department at an elementary school wishes to find out whether there is a need to implement a new reading programme. The department gathers information on the level of reading activity of each child in a random sample of 205 children and used school records to categorise them according to their literacy skills. The results of this analysis are shown in the table.

		Level of Reading Activity		
		High	Medium	Low
Literacy Skills	Good	25	12	7
	Average	35	57	27
	Poor	9	13	20

Carry out a chi-squared test for the independence of the two factors, level of reading activity and literacy skills. [6]

Discuss what the test indicates about the association, if any, between the two factors, and identify any issues that the academic department should be concerned about after this analysis. You should refer to the ***p*** value for your test and to the contributions of individual cells to the test statistic. [3]

[Turn over

8 The probability density function (pdf) of the random variable X

$$\mathrm{f}(x) = \frac{\gamma}{\pi}\left(\frac{1}{(x-\alpha)^2+\gamma^2}\right),\ x \in \mathbb{R},$$

follows a Cauchy distribution characterized by the constants $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$.
(α is also known as the location parameter and γ the scale parameter)

(a) State the median of X, giving a reason for your answer. [2]

(b) Show that $\mathrm{P}(X > \alpha + \gamma) = 0.25$.

[You may use the result $\lim\limits_{N\to+\infty} \tan^{-1} N = \frac{\pi}{2}$.]

Hence deduce the interquartile range. [3]

When $\alpha = 0$ and $\gamma = 1$, X is said to have a standard Cauchy distribution.

(c) On the same diagram, sketch the pdf of the standard Cauchy random variable and the standard Normal random variable, stating the relevant coordinates of any points of intersection and axial intercepts. [3]

The random variable U has a continuous uniform distribution on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the random variable C is defined by $C = \tan U$.

(d) Find the pdf of C and deduce that C has a Cauchy distribution where α and γ are values to be determined. [4]

9 **(i)** X and Y are independent random variables having Poisson distributions with means λ and μ respectively. By considering

$$\mathrm{P}(X+Y=r)=\sum_{k=0}^{r}\mathrm{P}(X=k)\mathrm{P}(Y=r-k),\ r=0,\ 1,\ 2,\ \ldots$$

show that $X+Y$ follows a Poisson distribution with mean $\lambda+\mu$. [3]

In the 2021 statistics report[1] released by the Singapore Civil Defence Force (SCDF), it was stated that SCDF responded to 585 Emergency Medical Services (EMS) calls a day.

(ii) State the conditions under which the number of EMS calls made per day can be modelled by a Poisson distribution. [2]

You should now assume these conditions hold.

(iii) Calculate the probability that, in a randomly chosen hour, more than 30 EMS calls will be made. [2]

In the same report, it was stated that SCDF responded to 5 Fire calls a day.

You should now assume that the number of Fire calls made also follows a Poisson distribution.

(iv) Calculate the probability that, in a randomly chosen 7-day week, exactly 50 Fire calls are made given that the number of EMS and Fire calls made are exactly 5000. State an assumption required for your calculations to be valid. [4]

(v) Find the average time interval, in seconds, between EMS calls made. [1]

[1]http://www.scdf.gov.sg/home/about-us/media-room/statistics

[Turn over

10 **(i)** Under what circumstances should a non-parametric test of a hypothesis be used in preference to a parametric test? [2]

The World Health Organisation recommends reducing sugar intake for better health as high sugar intake is linked to an increased risk of obesity and diabetes.

The Ministry Of Health, together with the Health Promotion Board, introduced the Nutri-Grade mark to help Singaporeans make informed choices across beverages. Packaged drinks, in particular, contributed to more than half of Singaporeans' daily sugar intake.

The Nutri-Grade mark consists of 4 grades (A, B, C and D) with Grade A corresponding to the lowest sugar and saturated fat content and Grade D corresponding to the highest sugar and saturated fat content. (See **Table 1** below for details regarding the sugar content).

Nutri-Grade mark	**A**	**B**	**C**	**D**
Sugar Content, x (grams per 100 ml)	$x \leq 1$ and no sweetener	$1 < x \leq 5$	$5 < x \leq 10$	$x > 10$

Table 1

John wants to investigate if the median sugar content of packaged drinks sold in his neighbourhood supermarket is less than 5 grams per 100 ml. He selected a random sample of 10 packaged drinks and recorded their Nutri-Grade marks, as shown below in **Table 2**. He made the assumption that Grades A/B represent less than 5 grams per 100 ml of sugar content and Grades C/D represent more than 5 grams per 100 ml of sugar content.

Drink	1	2	3	4	5	6	7	8	9	10
Nutri-Grade mark	B	B	A	C	D	B	B	B	D	B

Table 2

(ii) Explain why John should use a sign test and not a Wilcoxon test to carry out his investigation. [1]

(iii) Carry out a sign test at the 5% level of significance using the data in **Table 2** and state the conclusion John should reach. [4]

Mark tells John that his test would have been more accurate if he had recorded the actual sugar content of the 10 packaged drinks as the grading is dependent on not just the sugar content but also the saturated fat content.

Mark decides to look at the nutrition information on each of the 10 packaged drinks and tabulates **Table 3** below.

Drink	1	2	3	4	5	6	7	8	9	10
Sugar Content (grams per 100 ml)	4.6	2.3	0.7	5.9	10.3	3.7	4.9	4.8	10.2	3.9

Table 3

(iv) By comparing the sugar content of each drink to a reference drink containing 5 grams per 100 ml of sugar, conduct a 1-tail Wilcoxon test at the 5% level of significance using the data in **Table 3** and comment on the relative merits of the two tests used. [5]

End of Paper