2024 H2 MATH (9758/02) JC 2 PRELIMINARY EXAMINATION – SUGGESTED SOLUTIONS

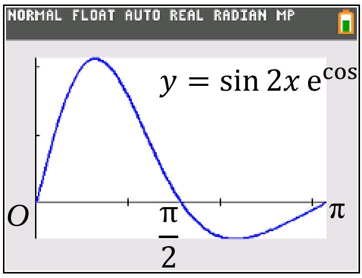
Qn	Solution
1	**Maclaurin’s Series**
(i)	$y=\ln\left(1+3x+2x^2\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3+4x}{1+3x+2x^2}$ Useful technique: Multiply both sides by the denominator before further differentiation. **AVOID** using quotient rule for further differentiation as much as possible. $\left(1+3x+2x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=3+4x$ diff wrt *x*, $\left(1+3x+2x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(3+4x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=4 \quad (\text{shown})$ Use implicit differentiation diff wrt *x*, $\left(1+3x+2x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(3+4x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+\left(3+4x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $\left(1+3x+2x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(6+8x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}=0$ Use implicit differentiation when $x=0,\ \ y=0,\ \ \frac{\mathrm{d}y}{\mathrm{d}x}=3,\ \ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-5,\ \ \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=18,$ Find the values of f(0), f’(0), f”(0) and f ’” (0). $y=3x-\frac{5}{2!}x^2+\frac{18}{3!}x^3+...$ $\quad=3x-\frac{5}{2}x^2+3x^3+...$ Substitute into the general form of the Maclaurin Series that can be found in MF26
(ii)	$y=\ln\left(1+3x+2x^2\right)$ $\quad=3x+2x^2-\frac{\left(3x+2x^2\right)^2}{2}+\frac{\left(3x+2x^2\right)^3}{3}+...$ Use the standard series in MF26 for $\ln\left(1+x\right)$ up to and including the term in x^3 and show clearly that it is the same as the Maclaurin Series found in **(i)**. $\quad=3x+2x^2-\frac{1}{2}\left(9x^2+12x^3+4x^4\right)+\frac{1}{3}\left(27x^3+...\right)$ $\quad=3x-\frac{5}{2}x^2+3x^3+...$ (verified) **<u>Alternative:</u>** $y=\ln\left(1+3x+2x^2\right)=\ln\left[\left(1+x\right)\left(1+2x\right)\right]$ $y=\ln\left(1+x\right)+\ln\left(1+2x\right)$ Using the standard series in MF26, $y=x-\frac{x^2}{2}+\frac{x^3}{3}+...+\left(2x-\frac{\left(2x\right)^2}{2}+\frac{\left(2x\right)^3}{3}\right)$ $\quad=3x-\frac{5}{2}x^2+3x^3+...$ (verified)

(iii)	$\ln\left(1+3x+2x^2\right)=3x-\frac{5}{2}x^2+3x^3+\ldots$ Sub $x=\frac{1}{2}$, $\ln\left(1+3\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)^2\right)=3\left(\frac{1}{2}\right)-\frac{5}{2}\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+\ldots$ $\therefore \ln 3\approx\frac{5}{4}$ Substitute $x=\frac{1}{2}$ into both sides of the Maclurin Series to get the approximate value for ln 3

Qn	Solution
2	**Vectors**
(i)	$\overrightarrow{OC}=-\frac{3}{2}\mathbf{a}$ $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are in opposite directions, thus the negative sign is important By ratio theorem, $\overrightarrow{OD}=\frac{4}{5}\mathbf{a}+\frac{1}{5}\mathbf{b}$ Area of triangle *OCD* $=\frac{1}{2}\left\lvert\overrightarrow{OC}\times\overrightarrow{OD}\right\rvert$ $=\frac{1}{2}\left\lvert-\frac{3}{2}\mathbf{a}\times\left(\frac{4}{5}\mathbf{a}+\frac{1}{5}\mathbf{b}\right)\right\rvert$ $=\frac{3}{4}\left\lvert\mathbf{a}\times\left(\frac{4}{5}\mathbf{a}+\frac{1}{5}\mathbf{b}\right)\right\rvert$ $=\frac{3}{4}\left\lvert\frac{4}{5}\mathbf{a}\times\mathbf{a}+\frac{1}{5}\mathbf{a}\times\mathbf{b}\right\rvert$ This is a SHOW question, there is a need to apply the distributive law and show the expansion clearly $=\frac{3}{4}\left\lvert\mathbf{0}+\frac{1}{5}\mathbf{a}\times\mathbf{b}\right\rvert$ Must explain that $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ $=\frac{3}{20}\lvert\mathbf{a}\times\mathbf{b}\rvert$ (shown)
(ii)	$\overrightarrow{AP}=\left(\frac{\overrightarrow{AO}\bullet\overrightarrow{AB}}{\lvert\overrightarrow{AB}\rvert}\right)\frac{\overrightarrow{AB}}{\lvert\overrightarrow{AB}\rvert}$ For the projection vector formula, there is NO modulus for $\frac{\overrightarrow{AO}\bullet\overrightarrow{AB}}{\lvert\overrightarrow{AB}\rvert}$ as the direction is important $=\left(\frac{-\mathbf{a}\bullet(\mathbf{b}-\mathbf{a})}{\lvert\mathbf{b}-\mathbf{a}\rvert}\right)\frac{(\mathbf{b}-\mathbf{a})}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $=\left(\frac{-\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{a}}{(\mathbf{b}-\mathbf{a})\bullet(\mathbf{b}-\mathbf{a})}\right)(\mathbf{b}-\mathbf{a})$ $\lvert\mathbf{a}\rvert^2=\mathbf{a}\bullet\mathbf{a}$ $\Rightarrow\lvert\mathbf{b}-\mathbf{a}\rvert^2=(\mathbf{b}-\mathbf{a})\bullet(\mathbf{b}-\mathbf{a})$ $=\left(\frac{-\mathbf{a}\bullet\mathbf{b}+\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{b}\rvert^2-2\mathbf{a}\bullet\mathbf{b}+\lvert\mathbf{a}\rvert^2}\right)(\mathbf{b}-\mathbf{a})\quad\left(\because\mathbf{a}\bullet\mathbf{a}=\lvert\mathbf{a}\rvert^2,\ \mathbf{b}\bullet\mathbf{b}=\lvert\mathbf{b}\rvert^2\right)$ $=\frac{\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2}(\mathbf{b}-\mathbf{a})\quad(\because\mathbf{a}\bullet\mathbf{b}=0)$ $\mu=\frac{\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2}$

Qn	Solution
3	**Complex Numbers**
(a)(i)	**NOTE: Do not use graphing calculator when answering this question.** **Answers obtained using GC will not be awarded marks.** Since coefficients of polynomial are **all** real, $z=2+\sqrt{3}\mathrm{i}$ is a root $\Rightarrow$ $z=2-\sqrt{3}\mathrm{i}$ is also a root. State Conjugate Root Theorem properly Quadratic factor $=\left(z-\left(2+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(2-\sqrt{3}\mathrm{i}\right)\right)$ $=\left((z-2)-\sqrt{3}\mathrm{i}\right)\left((z-2)+\sqrt{3}\mathrm{i}\right)$ $=(z-2)^2-(\sqrt{3}\mathrm{i})^2$ $=z^2-4z+7$ Let $z=b$ be the third root. (Note that the last root must be real.) $z^3-8z^2+23z+k=\left(z^2-4z+7\right)\left(z-b\right)$ Comparing coefficient of z^2: $-b-4=-8 \quad \Rightarrow \quad b=4$ Comparing coefficient of z^0: $k=7\left(-4\right)=-28$ Thus, the roots are $2+\sqrt{3}\mathrm{i}$, $2-\sqrt{3}\mathrm{i}$ and 4. Answer the question **<u>Alternative Method (Not recommended)</u>** $z^3-8z^2+23z+k==0$ Substitute $2+\sqrt{3}\mathrm{i}$ into equation: $\left(2+\sqrt{3}\mathrm{i}\right)^3-8\left(2+\sqrt{3}\mathrm{i}\right)^2+23\left(2+\sqrt{3}\mathrm{i}\right)+k=0$ $\left(8+12\sqrt{3}\mathrm{i}+6\left(\sqrt{3}\mathrm{i}\right)^2+\left(\sqrt{3}\mathrm{i}\right)^3\right)-8\left(4+4\sqrt{3}\mathrm{i}-\sqrt{3}^2\right)+46+23\sqrt{3}\mathrm{i}+k=0$ GC is not allowed so working needs to be shown clearly $8+12\sqrt{3}\mathrm{i}-18-3\sqrt{3}\mathrm{i}-32-32\sqrt{3}\mathrm{i}+24+46+23\sqrt{3}\mathrm{i}+k=0$ $28+k=0$ $k=-28$ No mark if working is not shown clearly Since coefficients of polynomial are **all** real, $z=2+\sqrt{3}\mathrm{i}$ is a root $\Rightarrow$ $z=2-\sqrt{3}\mathrm{i}$ is also a root. State Conjugate Root Theorem properly Quadratic factor $=\left(z-\left(2+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(2-\sqrt{3}\mathrm{i}\right)\right)$ $=\left((z-2)-\sqrt{3}\mathrm{i}\right)\left((z-2)+\sqrt{3}\mathrm{i}\right)$ $=(z-2)^2-(\sqrt{3}\mathrm{i})^2$ $=z^2-4z+7$ Let $z=b$ be the third root. (Note that the last root must be real.)

	$z^3-8z^2+23z-28=\left(z^2-4z+7\right)\left(z-b\right)$ Comparing constant $-b-4=-8 \quad \Rightarrow \quad b=4$ Thus, the roots are $2+\sqrt{3}\mathrm{i},\ 2-\sqrt{3}\mathrm{i}$ and 4. Answer the question
(a)(ii)	Replace z with $\mathrm{i}z$ $\mathrm{i}z=2+\sqrt{3}\mathrm{i}$ or $\mathrm{i}z=2-\sqrt{3}\mathrm{i}$ or $\mathrm{i}z=4$ $z=\sqrt{3}-2\mathrm{i}$ or $z=\sqrt{3}-2\mathrm{i}$ or $z=-4\mathrm{i}$ “Hence” so use previous part’s answer and do a replacement
(b)	$w=\cos\theta+\mathrm{i}\sin\theta=\mathrm{e}^{\mathrm{i}\theta}$ Tip: Since this involves division of 2 complex numbers, use exponential form $\frac{w^*}{w^2+1}=\frac{\mathrm{e}^{\mathrm{i}(-\theta)}}{\mathrm{e}^{\mathrm{i}(2\theta)}+1}$ $=\frac{\mathrm{e}^{\mathrm{i}(-\theta)}}{\mathrm{e}^{\mathrm{i}(\theta)}\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}(-\theta)}\right)}$ Useful result: $\mathrm{e}^{\mathrm{i}(2\theta)}=\mathrm{e}^{\mathrm{i}(\theta)}\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}(-\theta)}\right)$ $=\frac{\mathrm{e}^{\mathrm{i}(-2\theta)}}{2\cos\theta}$ $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}(-\theta)}=\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta=2\cos\theta$ $=\frac{1}{2}\sec\left(\theta\right)\mathrm{e}^{\mathrm{i}(-2\theta)}$ This is of the form $R\mathrm{e}^{\mathrm{i}(-2\theta)}$ where $R=\frac{1}{2}\sec\theta$ is the modulus and -2θ is the argument $\left\lvert\frac{w^*}{w^2+1}\right\rvert=\frac{1}{2}\sec\theta$ $\arg\left(\frac{w^*}{w^2+1}\right)=-2\theta$

Qn	**Solution**
4	**Integration**
(a)(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\cos x}\right) = -\sin x\mathrm{e}^{\cos x}$
(ii)	**Note:** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\cos x}\right) = -\sin x\,\mathrm{e}^{\cos x} \Rightarrow \int \sin x\,\mathrm{e}^{\cos x}\mathrm{d}x = -\mathrm{e}^{\cos x} + C$ For integration questions, if question asks you to differentiate something first, it is to guide you to see the integration Double angle formula (MF26): $\sin 2A = 2\sin A\cos A$ $\int \sin 2x\,\mathrm{e}^{\cos x}\,\mathrm{d}x$ $= \int 2\sin x\cos x\,\mathrm{e}^{\cos x}\,\mathrm{d}x$ $= 2\int \cos x\left(\sin x\,\mathrm{e}^{\cos x}\right)\mathrm{d}x$ K I – D I $u = \cos x \qquad v = (\sin x)\mathrm{e}^{\cos x}$ $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \qquad \int v\,\mathrm{d}x = -\mathrm{e}^{\cos x}$ $= 2\left[\cos x\left(-\mathrm{e}^{\cos x}\right) - \int(-\sin x)\left(-\mathrm{e}^{\cos x}\right)\mathrm{d}x\right]$ $= 2\left[-\cos x\,\mathrm{e}^{\cos x} - \int \sin x\,\mathrm{e}^{\cos x}\mathrm{d}x\right]$ $= 2\left[-\cos x\,\mathrm{e}^{\cos x} + \mathrm{e}^{\cos x}\right] + C$ $= 2\mathrm{e}^{\cos x}(1-\cos x) + C \qquad$ (shown) For integration by parts, **Highly recommended** to work out the u and v at the side, then apply KI – DI to fill in each part accordingly
(iii)	$\int_0^{\pi}\left\|\sin 2x\,\mathrm{e}^{\cos x}\right\|\mathrm{d}x$ **Keyword: exact** ⇒Cannot just use G.C. Need to remove the modulus before you can integrate From the graph of $y = \sin 2x\,\mathrm{e}^{\cos x}$, observe that when $y \le 0$, $\frac{\pi}{2} \le x \le \pi$ $\therefore \left\|\sin 2x\,\mathrm{e}^{\cos x}\right\| = \begin{cases} \sin 2x\,\mathrm{e}^{\cos x} & , 0 \le x < \frac{\pi}{2} \\ -\sin 2x\,\mathrm{e}^{\cos x} & , \frac{\pi}{2} \le x \le \pi \end{cases}$ NORMAL FLOAT AUTO REAL RADIAN MP $= \int_0^{\frac{\pi}{2}} \sin 2x\,\mathrm{e}^{\cos x}\,\mathrm{d}x - \int_{\frac{\pi}{2}}^{\pi} \sin 2x\,\mathrm{e}^{\cos x}\,\mathrm{d}x$ Use the show result from **(ii)** $= \left[2\mathrm{e}^{\cos x}(1-\cos x)\right]_0^{\frac{\pi}{2}} - \left[2\mathrm{e}^{\cos x}(1-\cos x)\right]_{\frac{\pi}{2}}^{\pi}$ $= 2\mathrm{e}^{0}(1-0) - 2\mathrm{e}^{1}(1-1) - \left[2\mathrm{e}^{-1}(1+1) - 2\mathrm{e}^{0}(1-0)\right]$ $= 4 - 4\mathrm{e}^{-1}$

(b)

$$x=\sec\theta \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec\theta\tan\theta$$

MF26

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$$

$$=\int \frac{1}{\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,\mathrm{d}\theta$$

Recall: $\tan^2\theta+1=\sec^2\theta$

$$=\int \frac{1}{\sqrt{\tan^2\theta}}(\sec\theta\tan\theta)\,\mathrm{d}\theta$$

$$=\int \frac{1}{\tan\theta}(\sec\theta\tan\theta)\,\mathrm{d}\theta \quad \text{since } 0<\theta<\frac{\pi}{2}$$

$$=\int \sec\theta\,\mathrm{d}\theta$$

$$=\ln\left|\sec\theta+\tan\theta\right|+C$$

MF26

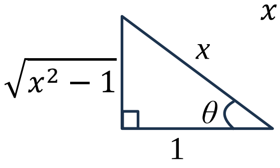
You should remove the modulus when possible

$$=\ln(\sec\theta+\tan\theta)+C \quad \text{since } 0<\theta<\frac{\pi}{2} \Rightarrow \sec\theta>0, \tan\theta>0$$

$$=\ln\left(x+\sqrt{x^2-1}\right)+C$$

Since it is indefinite integral, need to change back to *x* and remember the '+ *C*'

$$x=\sec\theta \Rightarrow \cos\theta=\frac{1}{x}$$



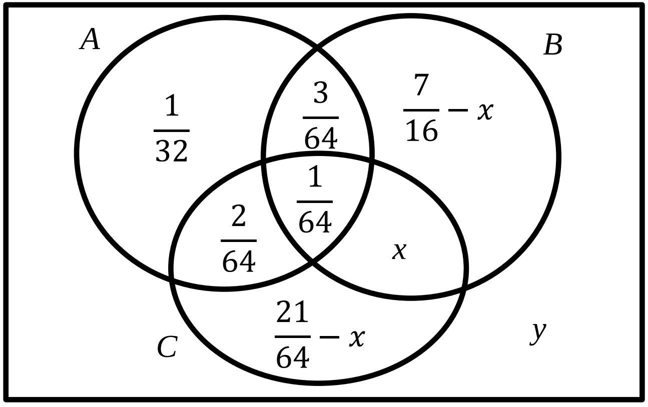
$$\therefore \tan\theta=\frac{\sqrt{x^2-1}}{1}$$

__Steps for integration by substitution:__

1) Differentiate the given substitution
   - Hidden working:

$$\underbrace{(\_\_)\,\mathrm{d}x}_{\text{Everything involving } x}=\underbrace{(\_\_)\,\mathrm{d}\theta}_{\text{Everything involving } \theta}$$

2) Substitute everything to θ in the integration in 1 step:
   - Expression (including the d*x*)
   - Upper & Lower limits (if any)
3) Simplify and integrate accordingly
4) For indefinite integrals, substitute back the *x* accordingly

Qn	Solution
5	**Probability**
(i)	$\mathrm{P}(B)=\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B')$ $=\frac{9}{16}-\frac{1}{16}$ $=\frac{8}{16}=\frac{1}{2}$
(ii)	$\mathrm{P}(A'\cap B)=\frac{1}{2}\left(1-\frac{1}{8}\right)=\frac{7}{16}$ $\mathrm{P}(A\cap B)=\frac{1}{2}-\frac{7}{16}=\frac{1}{16}$ and $\mathrm{P}(A)\mathrm{P}(B)=\frac{1}{8}\left(\frac{1}{2}\right)=\frac{1}{16}$ Since $\mathrm{P}(A)\mathrm{P}(B)=\mathrm{P}(A\cap B)=\frac{1}{16}$, the events *A* and *B* are independent. **Alternatively** Given that $\mathrm{P}(A\cap B')=\frac{1}{16}$ and $\mathrm{P}(A)\times\mathrm{P}(B')=\frac{1}{8}\times\frac{1}{2}=\frac{1}{16}$ Since $\mathrm{P}(A)\mathrm{P}(B')=\mathrm{P}(A\cap B')=\frac{1}{16}$, the events *A* and *B’* are independent. $\therefore$ the events *A* and *B* are independent.
(iii)	Since *C* is independent of *A*, $\therefore$ the events of A' and *C* are independent. $\mathrm{P}(A'\cap C)=\mathrm{P}(A')\times\mathrm{P}(C)$ $=\left(1-\frac{1}{8}\right)\left(\frac{3}{8}\right)$ $=\frac{21}{64}$
(iv)	Since *A* and *B* are independent, *A*’ and *B* are also independent, $\mathrm{P}(A'\cap B)=\mathrm{P}(A')\mathrm{P}(B)=\frac{7}{8}\left(\frac{1}{2}\right)=\frac{7}{16}$ Let $\mathrm{P}((B\cap C)\cap A')=x$ and $\mathrm{P}(A'\cap B'\cap C')=y$  From Venn diagram, $\frac{1}{32}+\frac{3}{64}+\frac{1}{64}+\frac{2}{64}+\frac{7}{16}-x+x+\frac{21}{64}-x+y=1$ $\frac{57}{64}-x+y=1$

	$y=\frac{7}{64}+x$ For minimum *y*, minimum $x=0$, $\Rightarrow y=\frac{7}{64}$ For maximum *y*, maximum $x=\frac{21}{64}$, $\Rightarrow y=\frac{7}{64}+\frac{21}{64}=\frac{7}{16}$ $\therefore \frac{7}{64}\le \mathrm{P}\left(A'\cap B'\cap C'\right)\le \frac{7}{16}$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>6</td><td>DRV</td></tr>
<tr><td>(a)(i)</td><td>
$p+2p+5p=1$
Total sum of probabilities = 1
$p=\frac{1}{8}$
$\mathrm{P}(X=6)=\left(\frac{5}{8}\right)\left(\frac{1}{8}\right)2!+\left(\frac{2}{8}\right)^2=\frac{7}{32}.$
Case 1: 1 Bull’s eye and 1 outer ring
$\left(\frac{5}{8}\right)\left(\frac{1}{8}\right)2!$
No of cases = No of ways to permute 1 Bull’s eye and 1 outer ring
Case 2: 2 inner ring
$\left(\frac{2}{8}\right)^2$ There is only one case.
$\mathrm{P}(X=2)=\left(\frac{5}{8}\right)^2=\frac{25}{64}.$
$\mathrm{P}(X=4)=\left(\frac{5}{8}\right)\left(\frac{2}{8}\right)2!=\frac{5}{16}.$
$\mathrm{P}(X=8)=\left(\frac{1}{8}\right)\left(\frac{2}{8}\right)2!=\frac{1}{16}.$
$\mathrm{P}(X=10)=\left(\frac{1}{8}\right)^2=\frac{1}{64}.$
<table>
<tr><td>x</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td></tr>
<tr><td>$\mathrm{P}(X=x)$</td><td>$\frac{25}{64}$</td><td>$\frac{5}{16}$</td><td>$\frac{7}{32}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{64}$</td></tr>
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</td></tr>
<tr><td>(ii)</td><td>Using GC, $\mathrm{E}(X)=4$</td></tr>
<tr><td>(iii)</td><td>
For the game to be fair, $\mathrm{E}\left(\frac{X}{2}-k\right)=0\Rightarrow\underbrace{\frac{1}{2}\mathrm{E}(X)}_{\text{Expected payout}}-\underbrace{k}_{\text{Cost of 1 round}}=0\Rightarrow k=2.$
<u>Idea:</u> Fair game refers to no loss or profit from playing the game.
Hence, expected payout (expectation) $-$ cost of 1 round $=0$
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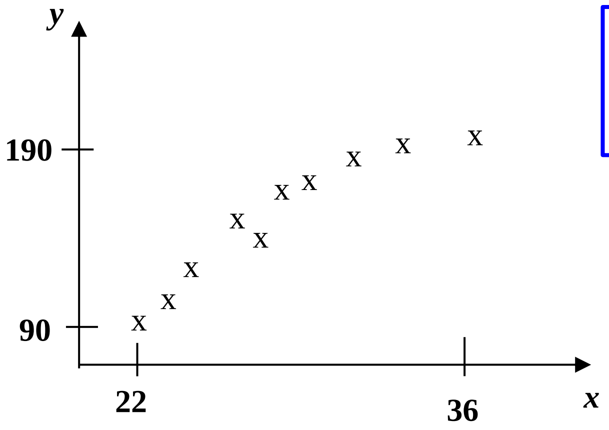
Qn	Solution
7	**Correlation and Regression**
(i)	 Only some did not label the max and min values on scatter diagram. From the scatter diagram, as the **daily average temperature increases**, the **number of popsicles sold increases at a decreasing rate**. (or as ***x* increases, *y* increases at a decreasing rate)** Note: Comment the relationship from scatter diagram instead of using *r* value.
(ii)	For $y=a+bx$, r value = 0.95143 = 0.951 (3 s.f.) For $\ln(200-y)=c+dx$, r value = $-0.98920=-0.989$ (3s.f.) Since $\lvert r\rvert=0.989$ for the model $\ln(200-y)=c+dx$ is closer to 1 than the $\lvert r\rvert=0.951$ for the model $y=a+bx$, hence $\ln(200-y)=c+dx$ is a better model. **OR** Since the product moment correlation coefficient for the model $\ln(200-y)=c+dx$ of -0.989 is closer to -1 than the product moment correlation coefficient for the model $y=a+bx$ of 0.951 to 1, hence $\ln(200-y)=c+dx$ is a better model. $\ln(200-y)=8.7493-0.18032x=8.75-0.180x$ Need to compare the |*r*| value of both models or *r* values of both models.
(iii)	When $x=37$, $y=192$ (3 s.f.) The estimate is not reliable as $x=37$ is outside the data range $22\le x\le 36$ and a linear model between $\ln(200-y)$ and x may not hold. Need to justify that the value of 37 is outside the data range.
(iv)	The value of 200 is the theoretical maximum number of popsicles that the store can sell per day. **Keywords: theoretical maximum number** of popsicles sold or the number of popsicles sold will **approach 200**.

Qn	Solution	
8	**Hypothesis Testing**	**Comments**
(i)	Let X be the mass of a randomly chosen orange (in grams). Let μ denote the population mean mass of an orange (in grams).	Define the variables accurately.
	$s^2=\frac{40}{40-1}\left(4^2\right)=\frac{640}{39}=16.410$ $\overline{x}=364.2$	4 is sample standard deviation. Use the formula $s^2=\frac{n}{n-1}\times\text{sample variance}$ to find s^2.
	$\text{H}_0:\mu=365$ $\text{H}_1:\mu<365$	The fruit seller claims that the mean mass is 365g. To test if he has overstated the mean, we test if the mean is less than 365g.
	Under H_0, since $n=40$ is large, by Central Limit Theorem, $\overline{X}\sim \text{N}\left(365,\frac{\left(\frac{640}{39}\right)}{40}\right)$ approximately.	Since the distribution of X is unknown and the sample size is large, we can use Central Limit Theorem to find the distribution of $\overline{X}$. Ensure phrasing is complete and CLT is spelt out in full.
	Test statistic: $Z=\frac{\overline{X}-365}{\sqrt{\frac{\left(640/39\right)}{40}}}\sim \text{N}\left(0,1\right)$	Standardise $\overline{X}$ to obtain the test statistic.
	Level of significance: 5% Reject H_0 if p-value < 0.05	Write down the level of significance and the rejection criteria. We can use p-value since there are no unknowns in the question.
	Under H_0, using G.C., $p\text{-value}=0.10583$ Since $p\text{-value}=0.106>0.05$, we do not reject H_0 and conclude that there is insufficient evidence at 5% level, that the population mean mass of the orange is less than 365g. Hence, the fruit seller has not overstated the mean at 5% level.	Take note of what is necessary in the conclusion: 1. Reject/do not reject H_0 2. Level of significance 3. Sufficient/insufficient evidence for H_1 (written in context) The question is whether the mean is overstated, not whether the fruit seller's claim is valid. Answer the question.

(ii)

Let D be the diameter of a randomly chosen orange (in cm).
Let μ denote the population mean diameter of an orange (in cm).

Define the variables accurately. Note that D is defined in the question and does not have to be redefined.

$H_0 : \mu = 9$

$H_1 : \mu > 9$

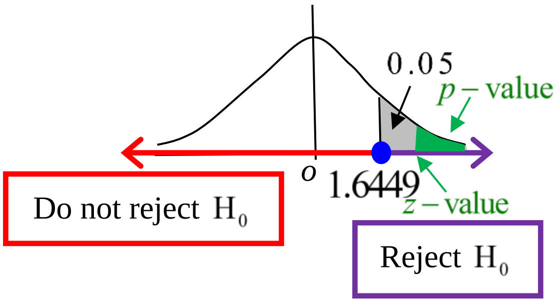
'Expected value of D is 9' means that 'the mean of D is 9'. In particular, we are testing if the mean diameter exceeds 9.

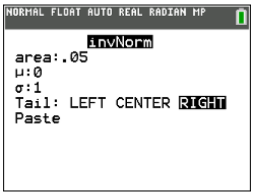
Under H₀, $D \sim \mathrm{N}\left(9, 0.3^2\right)$, $\therefore \overline{D} \sim \mathrm{N}\left(9, \frac{0.3^2}{50}\right)$.

Test Statistic: $Z = \dfrac{\overline{D} - 9}{\sqrt{\dfrac{0.3^2}{50}}} \sim \mathrm{N}(0,1)$

Level of significance: 5%

Write down the reason and the distribution of $\overline{D}$ clearly. It is stated that D is normally distributed and the population standard deviation is given to be 0.3.





invNorm
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μ:0
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Reject H_0 if z-value > 1.6449

Write down the rejection criteria for the right-tail test at 5% level of significance. We have to use z-value method as sample mean is unknown. Represent it using $\bar{d}$.

Let $\bar{d}$ denote the sample mean diameter of an orange for this sample.

$$z\text{-value} = \frac{\bar{d} - 9}{\left(0.3 \big/ \sqrt{50}\right)}$$

Since there is sufficient evidence to reject H₀,
$\therefore$ z-value > 1.6449

$$\frac{\bar{d} - 9}{\left(0.3 \big/ \sqrt{50}\right)} > 1.6449$$

$\bar{d} > 9.0698$

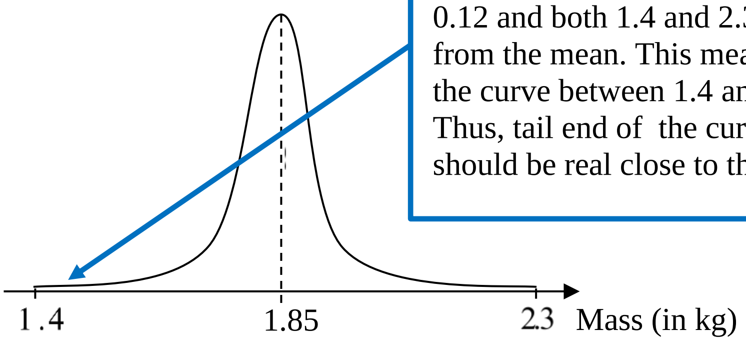
$\therefore \bar{d} > 9.07 \quad (3 \text{ s.f.})$

'Sufficient evident from the sample to conclude that the mean diameter exceeds 9cm' means that there is sufficient evidence for H₁ i.e. H₀ is rejected. Solve for the unknown $\bar{d}$. Round off the answer to satisfy the inequality.

(iii)	Since $\bar{d} = 8.9 < 9.07$ is outside the rejection region found in **(ii),** we do not reject H_0 and conclude that there is insufficient evidence, at 5% level of significance, that the population mean diameter of the orange is more than 9 cm. Take note of what is necessary in the conclusion: 1. Reject/do not reject H_0 2. Level of significance 3. Sufficient/insufficient evidence for H_1	Note that the hypotheses and sample size for **(ii)** and **(iii)** are the same. The population variance is also constant. Hence we can use the result in **(ii)**. From **(ii)**, for H_0 to be rejected, the sample mean is greater than 9.07. Since the sample mean is less than 9.07 for this part, we do not reject H_0. Ensure that clear reasoning is given and conclusion is complete.

Qn	Solution
9	**Binomial Distribution & Probability**
(i)	Assumptions: For Binomial Distribution, need to explain the following assumptions **in the context of the question** 1. Independent trials (NOT independent probability) 2. Constant probability of success The other 2 conditions (fixed number of trials and two mutually exclusive outcomes) are implied to be true from the question. 1. Whether a randomly chosen ball tossed by the participant goes into the container is **independent** of any other balls tossed. 2. The **probability** that a randomly chosen ball tossed by the participant goes into the container is **constant** at 0.35. The participant may adjust the toss after he succeeds or fails, hence the condition of independence may not hold. Reason given must explain clearly why an assumption may not hold by relating to a round of the game played by a participant.
(ii)	$X \sim \mathrm{B}(5, 0.35)$ $\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X \leq 2)$ $= 0.23517$ $= 0.235$ Important to use complement to change to $1 - \mathrm{P}(X \leq 2)$ so that GC Binomcdf command can be used
(iii)	Let Y be the number of participants, out of 10, who did not win a prize. $Y \sim \mathrm{B}(10, 1 - 0.23517) \Rightarrow Y \sim \mathrm{B}(10, 0.76483)$ $\mathrm{P}(Y \leq 6) = 0.191$ Use 5 s.f value **Important:** define all variables clearly **Altenative solution:** Let W be the number of participants, out of 10, who won a prize. $W \sim \mathrm{B}(10, 0.23517)$ $\mathrm{P}(W \geq 4) = 1 - \mathrm{P}(W \leq 3)$ $= 0.191$ At most 6 (out of 10) did not win a prize **is equivalent to** At least 4 (out of 10) won a prize

(iv)	$\mathrm{P}(\text{Wins a prize})$ $=\mathrm{P}(X\ge 3)+\mathrm{P}(X=2)\mathrm{P}(X\ge 4)$ $=(0.23517)+(0.33642)(0.054023)$ $=0.25334$ $=0.253$ Consider 2 cases for a participant to win a prize: **Case 1:** participant tosses at least 3 balls into the container in 1st round only **Case 2:** participant tosses exactly 2 balls into the container in 1st round **and** tosses at least 4 balls into the container in 2nd round.
(v)	$\mathrm{P}(\text{Toss fewer than 7 balls into the container} \mid \text{Wins a prize})$ $=\dfrac{\mathrm{P}(\text{Toss fewer than 7 balls into the container and wins a prize})}{\mathrm{P}(\text{Wins a prize})}$ $=\dfrac{\mathrm{P}(X\ge 3)+\mathrm{P}(X=2)\mathrm{P}(X=4)}{0.25334}$ $=\dfrac{0.25128}{0.25334}$ $=0.993$ Use conditional probability For the numerator, consider 2 cases for a participant to toss fewer than 7 balls and win a prize **Case 1:** participant tosses at least 3 balls into the container in 1st round only **Case 2:** participant tosses exactly 2 balls into the container in 1st round **and** tosses **exactly** 4 balls into the container in 2nd round. For the denominator, use the P(wins a prize) value (up to at least 5 s.f) found in **part (iv).**

Qn	Solution
10	**Normal and Sampling Distributions**
(i)	 Note that the standard deviation (SD) is 0.12 and both 1.4 and 2.3 is 3.75 SD away from the mean. This meant the area under the curve between 1.4 and 2.3 is close to 1. Thus, tail end of the curve at 1.4 and 2.3 should be real close to the axis.
(ii)	Let X be the mass of a randomly chosen papaya (in kg). $X \sim \mathrm{N}\left(1.85, 0.12^2\right)$ Let Y be the mass of a randomly chosen watermelon (in kg). $Y \sim \mathrm{N}\left(6.5, 0.72^2\right)$ $\mathrm{P}\left(X < 1.7\right) = 0.106 \quad (3 \text{ s.f.})$ **Note:** You must: a) Define the variables used clearly and b) Write down the distribution with the parameter values evaluated.
(iii)	$\mathrm{E}\left(Y - X\right) = 6.5 - 1.85 = 4.65$ $\mathrm{Var}\left(Y - X\right) = 0.72^2 + 0.12^2 = 0.5328$ $Y - X \sim \mathrm{N}\left(4.65, 0.5328\right)$ $\mathrm{P}\left(Y - X > 4.5\right) = 0.581 \ (3 \text{ s.f.})$ **Note:** a) $\mathrm{Var}\left(Y - X\right) = \mathrm{Var}\left(Y\right) + \mathrm{Var}\left(X\right)$ b) 0.5328 is an exact answer. Do NOT round off to 3 s.f.
(iv)	$\bar{Y} = \frac{Y_1 + Y_2 + \cdots Y_n}{n} \sim \mathrm{N}\left(6.5, \frac{0.72^2}{n}\right)$ $\mathrm{P}\left(\bar{Y} > k\right) = 0.2$ $\mathrm{P}\left(Z > \frac{k - 6.5}{0.72/\sqrt{n}}\right) = 0.2$ $\frac{k - 6.5}{0.72/\sqrt{n}} = 0.84162$ $k = 6.5 + \frac{0.606}{\sqrt{n}}$
(v)	Let $T = 2.20\left(X_1 + X_2 + X_3\right) + 1.45\left(Y_1 + Y_2 + Y_3 + Y_4\right)$ $\mathrm{E}\left(T\right) = 2.20 \times 3 \times 1.85 + 1.45 \times 4 \times 6.5 = 49.91$ $\mathrm{Var}\left(T\right) = 2.20^2 \times 3 \times 0.12^2 + 1.45^2 \times 4 \times 0.72^2 = 4.568832$ $T \sim N\left(49.91, 4.568832\right)$ $\mathrm{P}\left(T > 50\right) = 0.483 \ (3 \text{ s.f.})$ **Note**: Do NOT round off exact answer to 3 s.f. In this question, 49.91 and 4.568832 are both exact answers.
(vi)	We assume that the **distributions** of the **masses** of **all** fruits are **independent** of one another.

Note: The key words are in bold.
Common Mistake: Phrasings by students are not clear in illustrating the need of independence between the distribution of masses of any two randomly chosen papayas (watermelons).