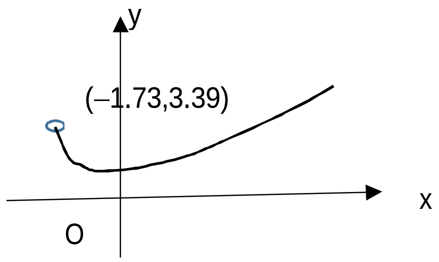
Jurong Pioneer Junior College

H2 Mathematics **JC1-2021**

End-of-Year Exam P1 solution

Question	Solution	Remarks
1(i)		Set min t=–2 in window setting Need to label and exclude the end-point
1(ii)	$\frac{dx}{dt}=1+2e^t, \frac{dy}{dt}=2-e^{-t}$ $\frac{dy}{dx}=\frac{2-e^{-t}}{1+2e^t}$ At P, $\frac{dy}{dx}=\frac{1}{3}$ Gradient of normal at P = –3 At P, x = 2, y = 1 The equation of normal is $y - 1 = -3(x - 2)$ $y = -3x + 7$	Note that $\frac{d}{dt}(e^t)=e^t$ Note that $e^0 = 1$
2(i)	(1, –3): $\frac{a}{4}+b+c=-3----(1)$ (–5,–21): $\frac{a}{-2}-5b+c=-21----(2)$ $\frac{dy}{dx}=-\frac{a}{(x+3)^2}+b$ At x = –5: $\frac{dy}{dx}=0 \Rightarrow -\frac{a}{4}+b=0---(3)$ Using GC: a = 8, b = 2, c = –7 So the equation of C is $y=\frac{8}{x+3}+2x-7$	Since there are 3 unknowns, we need 3 equations. Note that C has a stationary point (–5, –21) means that C (i) passes through (–5, –21); (ii) $\frac{dy}{dx}=0$ at x = –5
(ii)	The two equations of asymptotes are: $x = -3$ $y = 2x - 7$	Vertical asymptote: $x = -3$ Oblique asymptote: $y = bx + c$

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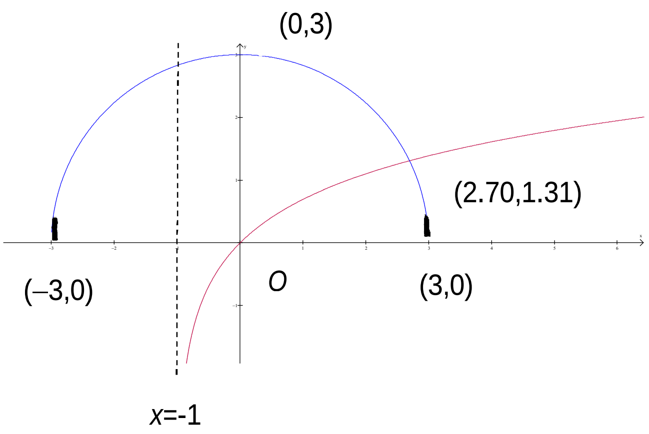
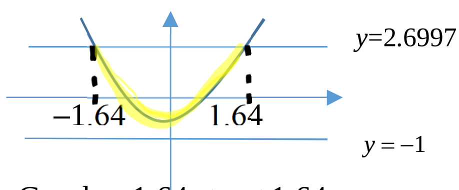
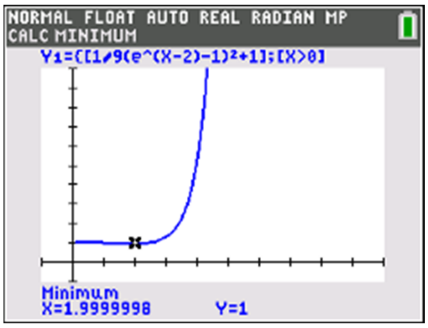
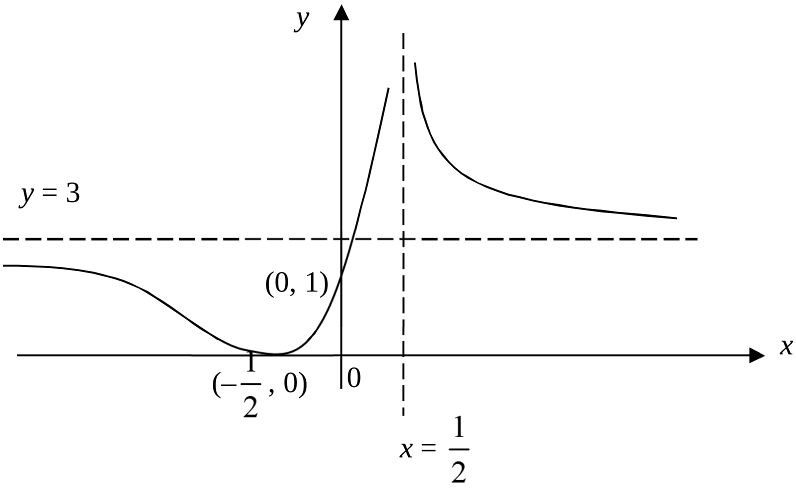
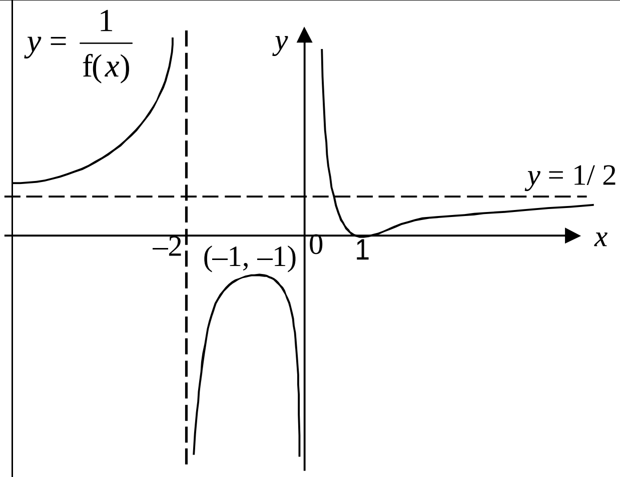
3	$x^2y-\tan^{-1}y=\frac{3}{4}\pi$ Differentiate wrt *x*, $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+y(2x)-\frac{1}{1+y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1}{1+y^2}-x^2\right)=2xy$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2xy}{\left(\frac{1}{1+y^2}-x^2\right)}=\frac{2xy\left(1+y^2\right)}{1-x^2-x^2y^2}$	
	when $y = 1$ $x^2(1)-\tan^{-1}1=\frac{3}{4}\pi$ $x^2-\frac{\pi}{4}=\frac{3}{4}\pi$ $x^2=\pi$ $x=\pm\sqrt{\pi}$ Since $x > 0$, $x=\sqrt{\pi}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\sqrt{\pi}\left(1+1^2\right)}{1-\pi-\pi(1)^2}=\frac{4\sqrt{\pi}}{1-2\pi}$ (exact)	We use degree only for vectors (or otherwise stated). Hence $\tan^{-1}1=\frac{\pi}{4}$ and not 45.
4(i)		The semi-circle must intersect with *x*-axis You need to indicate the vertical asymptote of the ln graph (GC does not show)
(ii)	The point of intersection is (2.70, 1.31) So the solution is $-1< x < 2.70$	
(iii)	Replace *x* by x^2: $-1< x^2 < 2.6997$ <u>Method 1</u>  Using Graph, $-1.64 < x < 1.64$	Proper method (in getting final answer) must be shown in order to gain full credit.

	Method 2 Since $x^2 \geq 0$, $-1 < x^2 < 2.6997$ can be simplified to $x^2 < 2.6997$ $(x - 1.64)(x + 1.64) < 0$ $-1.64 < x < 1.64$	It is incorrect to reject the part on $-1 < x^2$. Use of identity $a^2 - b^2 = (a + b)(a - b)$

Question	Solution	Remarks
5(i)	Let $y = \frac{1}{3}\left(e^{x-2} - 1\right)$ $e^{x-2} - 1 = 3y$ $x - 2 = \ln(3y + 1)$ $x = \ln(3y + 1) + 2$ $f^{-1}(x) = \ln(3x + 1) + 2$, $x > 0$	$D_{f^{-1}} = R_f$
(ii)	As the graph of f^{-1} intersects the graph of f in the line $y = x$, the solution of $f(x) = f^{-1}(x)$ is the same as the solution of $f(x) = x$. Hence $\frac{1}{3}\left(e^{x-2} - 1\right) = x$ $e^{x-2} = 3x + 1$ Use GC, the solution is $x = 4.72$	Clear explanation must be given as well as how $e^{x-2} = 3x + 1$ is obtained. Intersection point satisfies $x > 2$
(iii)	$R_f = (0, \infty)$ and $D_g = (-\infty, \infty)$ So $R_f \subseteq D_g$, gf exists $gf(x) = g\left(\frac{1}{3}\left(e^{x-2} - 1\right)\right) = \frac{1}{9}\left(e^{x-2} - 1\right)^2 + 1$, $x > 2$  $R_{gf} = (1, \infty)$	Justification of $R_f \subseteq D_g$ should be given (i.e. $R_f = (0, \infty)$ & $D_g = (-\infty, \infty)$ $D_{gf} = D_f$

Question	Solution	Remarks
6(a)	$\mathbf{u}\bullet\mathbf{v}=0 \Rightarrow \mathbf{u}\perp\mathbf{v}$ or $\mathbf{u}$ is a zero vector or $\mathbf{v}$ is a zero vector	Question does not indicate the vector is non-zero vector.
(b)(i)	Since N lies on l, $\overrightarrow{ON}=\frac{1}{3}\mathbf{a}+\lambda(\mathbf{2b}-\mathbf{a})$ for some λ Since AN is perpendicular to l, $\overrightarrow{AN}\bullet(2\mathbf{b}-\mathbf{a})=0$ $\left(\left(-\frac{2}{3}-\lambda\right)\mathbf{a}+2\lambda\mathbf{b}\right)\bullet(2\mathbf{b}-\mathbf{a})=0$ $\left(-\frac{2}{3}-\lambda\right)\mathbf{a}\bullet2\mathbf{b}+\left(\frac{2}{3}+\lambda\right)\mathbf{a}\bullet\mathbf{a}+4\lambda\mathbf{b}\bullet\mathbf{b}-\mathbf{2}\lambda\mathbf{a}\bullet\mathbf{b}=0$ Since **a** and **b** are perpendicular, $\therefore \mathbf{a}\bullet\mathbf{b}=0$ $\left(\frac{2}{3}+\lambda\right)\mathbf{a}\bullet\mathbf{a}+4\lambda\mathbf{b}\bullet\mathbf{b}=0$ $\left(\frac{2}{3}+\lambda\right)\lvert\mathbf{a}\rvert^2+4\lambda\lvert\mathbf{b}\rvert^2=0$ $\left(\frac{2}{3}+\lambda\right)(4)+4\lambda(1)=0$ $\lambda=-\frac{1}{3}$ $\overrightarrow{ON}=\frac{1}{3}\mathbf{a}-\frac{1}{3}(\mathbf{2b}-\mathbf{a})=\frac{2}{3}\mathbf{a}-\frac{2}{3}\mathbf{b}$	Use the concept that a point lies on a line, its position vector satisfies the equation of the line.
(ii)	Area of triangle OAN = $\frac{1}{2}\lvert\overrightarrow{OA}\times\overrightarrow{ON}\rvert$ $=\frac{1}{2}\left\lvert\mathbf{a}\times\left(\frac{2}{3}\mathbf{a}-\frac{2}{3}\mathbf{b}\right)\right\rvert$ $=\frac{1}{3}\lvert\mathbf{a}\times(\mathbf{a}-\mathbf{b})\rvert$ $=\frac{1}{3}\lvert\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}\rvert$ $=\frac{1}{3}\lvert\mathbf{0}-\mathbf{a}\times\mathbf{b}\rvert$ $=\frac{1}{3}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\lvert\sin 90^\circ\rvert$ $=\frac{1}{3}(2)(1)(1)$ $=\frac{2}{3}$	Apply area of triangle using cross product.

Question	**Solution**	**Remarks**
7a(i)	$y = \mathrm{f}\left(2x\right) + 1$ 	Note that $y = \mathrm{f}\left(2x\right)$ is a scaling parallel to the x-axis by a factor of $\frac{1}{2}$. Label all intercepts and asymptotes clearly
(ii)		Label all intercepts and asymptotes clearly. Note that the x-intercept at (1,0) should not be drawn as a 'sharp' point, it should be drawn like a min. point.
(b)	$y = \dfrac{1}{x+1} \xrightarrow{\text{A}} y = \dfrac{1}{-x+1} \xrightarrow{\text{B}} y = \dfrac{1}{-(x+5)+1} = \dfrac{1}{-(x+4)}$	Working must be shown clearly, step by step. Credit is not given even if your answer is correct, but you skipped step(s).

<table>
<tr><th>Question</th><th>Solution</th><th>Remarks</th></tr>
<tr><td>8(a)</td><td>

$\pi r^2 + 2\pi rh + \pi r(10) = 100\pi$

$h = \dfrac{100 - 10r - r^2}{2r}$

</td><td>Use info given by the question.</td></tr>
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$V = \dfrac{1}{3}\pi r^2 h + \pi r^2 h = \dfrac{4}{3}\pi r^2 \left(\dfrac{100 - r^2 - 10r}{2r}\right)$

$V = \dfrac{2}{3}\pi\left(100r - r^3 - 10r^2\right)$

$\dfrac{dV}{dr} = \dfrac{2}{3}\pi\left(100 - 3r^2 - 20r\right) = 0$

$-3r^2 - 20r + 100 = 0$

$(-3r + 10)(r + 10) = 0$

or

$r = \dfrac{20 \pm \sqrt{400 + 4(3)(100)}}{2(-3)} = \dfrac{20 \pm 40}{-6} = -10(\text{NA}), \dfrac{10}{3}$

Either 2nd derivative test:

$\dfrac{d^2V}{dr^2} = \dfrac{2}{3}\pi\left(-6r - 20\right) < 0$ since $r > 0$

or

$\dfrac{d^2V}{dr^2} = \dfrac{2}{3}\pi\left(-6\left(\dfrac{10}{3}\right) - 20\right) = -\dfrac{80}{3}\pi < 0$ when $r = \dfrac{10}{3}$

Or 1st derivative test

<table>
<tr><td>r</td><td>3</td><td>$\dfrac{10}{3}$</td><td>4</td></tr>
<tr><td>$\dfrac{dV}{dr}$</td><td>27.2</td><td>0</td><td>–58.6</td></tr>
<tr><td>tangent</td><td>/</td><td>-</td><td>\</td></tr>
</table>

V is max when $r = \dfrac{10}{3}$

$V = \dfrac{2}{3}\pi\left(\dfrac{10}{3}\right)\left(100 - \dfrac{100}{9} - \dfrac{100}{3}\right) = \dfrac{10000\pi}{81}$ or $123\dfrac{37}{81}\pi$

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Simplify the expression for V will make the differentiation much simpler.

No need to apply product or quotient rule.

Qn needs **“exact”** answer. Required to show working for finding r either by the quadratic formula for factorisation. and working for max V.

If using 2nd derivative test, either substitute in 10/3 to show that $\dfrac{d^2V}{dr^2} < 0$ or explain that when $r > 0$, $\dfrac{d^2V}{dr^2} < 0$

When using the 1st derivative test, numerical values of $\dfrac{dV}{dr}$ should be given. Not just the sign + /-

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Question	Solution	Remarks
8(b)	Volume of Cone $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi r^2(2r)=\frac{2}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r}=2\pi r^2$ Base area $A=\pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r}=2\pi r$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}=\left(2\pi r^2\right)\frac{\mathrm{d}r}{\mathrm{d}t}$ $15=2\pi(3)^2\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{5}{6\pi}$ $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}=\left(2\pi r\right)\frac{\mathrm{d}r}{\mathrm{d}t}=\left(2\pi\right)(3)\left(\frac{5}{6\pi}\right)=5\text{ cm}^2\text{ min}^{-1}$ Alternatively, $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}V}\frac{dV}{dt}=\left(2\pi r\right)\left(\frac{1}{2\pi r^2}\right)(15)=5\text{cm}^2\text{ min}^{-1}$	(b) is not related to (a), we only consider a cone. Note where the "π" is placed. $\frac{5}{6\pi}$, not $\frac{5}{6}\pi$ Note units of rate of change of area is <u>cm² / min</u>
9(i)	$\overrightarrow{PQ}=\begin{pmatrix}4\\2\\-1\end{pmatrix}-\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}3\\1\\-3\end{pmatrix}$ The length of projection $\dfrac{\left\|\begin{pmatrix}3\\1\\-3\end{pmatrix}\times\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{3}}$ $= =\frac{1}{\sqrt{3}}\left\|\begin{pmatrix}4\\-6\\2\end{pmatrix}\right\|$ $=\sqrt{\frac{56}{3}}=2\sqrt{\frac{14}{3}}=\frac{2}{3}\sqrt{42}$	Not scalar product formula Modulus is to be seen

Question	**Solution**	**Remarks**
9(ii)	Let *M* be the point of intersection. $\overrightarrow{OM}=\begin{pmatrix}1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-3\end{pmatrix}$ for some λ $\overrightarrow{OM}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=3\Rightarrow\begin{pmatrix}1+3\lambda\\1+\lambda\\2-3\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=3$ $\lambda+4=3 \quad\Rightarrow\quad \lambda=-1$ $\overrightarrow{OM}=\begin{pmatrix}1\\1\\2\end{pmatrix}+(-1)\begin{pmatrix}3\\1\\-3\end{pmatrix}=\begin{pmatrix}-2\\0\\5\end{pmatrix}$	Note notation for vector
(iii)	Use normal vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\mathbf{r}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\4\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=-2$ $x=-2$	Note the normal vector of y-z plane Cartesian equation asked
(iv)	$\begin{pmatrix}a\\7\\b\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=3\Rightarrow a+7+b=3$ $\begin{pmatrix}a\\7\\b\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=-2\Rightarrow a=-2$ $\therefore b=-2$	Position vector of a point lying on plane satisfies the equation of plane

Question	Solution	Remarks
9(v)	Direction vector of line of intersection $= \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$ Vector equation required is $\mathbf{r} = \begin{pmatrix}-2\\7\\-2\end{pmatrix} + \mu \begin{pmatrix}0\\1\\-1\end{pmatrix}, \ \mu \in \mathbb{R}$ Alternatively, Solve $x + y + z = 3$ and $x = -2$, Using GC, Obtain $\mathbf{r} = \begin{pmatrix}-2\\5\\0\end{pmatrix} + \mu \begin{pmatrix}0\\1\\-1\end{pmatrix}, \ \mu \in \mathbb{R}$	Find direction vector of line of intersection of planes Use GC to solve the Cartesian equations
10(a)	$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$ $= e^{\sin^{-1} x} + c$ since $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	Use of recognised form
(b)	$f(x) = \frac{x^2 - 6x + 1}{(3x+1)(x^2+3)}$ $= \frac{A}{3x+1} + \frac{Bx+C}{x^2+3}$ $= \frac{A(x^2+3) + (Bx+C)(3x+1)}{(3x+1)(x^2+3)}$ $x^2 - 6x + 1 = Ax^2 + 3A + 3Bx^2 + 3Cx + Bx + C$ $= (A+3B)x^2 + (3C+B)x + 3A + C$ Comparing coefficients, $A + 3B = 1$ $B + 3C = -6$ $3A + C = 1$ From GC, $A = 1$, $B = 0$, $C = -2$ $\therefore f(x) = \frac{x^2 - 6x + 1}{(3x+1)(x^2+3)} = \frac{1}{(3x+1)} - \frac{2}{(x^2+3)}$	Can use compare coefficients method, or substituting appropriate values or rearrangement of terms.

	Alternatively $\mathrm{f}(x)=\dfrac{x^2-6x+1}{(3x+1)(x^2+3)}=\dfrac{x^2+3-6x-2}{(3x+1)(x^2+3)}$ $=\dfrac{1}{3x+1}-\dfrac{2(3x+1)}{x^2+3}=\dfrac{1}{3x+1}-\dfrac{2}{x^2+3}$ So $A=1$, $B=0$, $C=-2$ Alternatively $A(x^2+3)+(Bx+C)(3x+1)=x^2-6x+1$ $x=-\dfrac{1}{3}: A\left(\dfrac{28}{9}\right)=\dfrac{1}{9}+2+1=\dfrac{28}{9}\Rightarrow A=1$ $x=0: 3+C=1\Rightarrow C=-2$ $x=1: 4+(B-2)(4)=-4\Rightarrow B=0$ $\int \mathrm{f}(x)\ \mathrm{d}x=\int\left[\dfrac{1}{3x+1}-\dfrac{2}{x^2+3}\right]\mathrm{d}x$ $=\dfrac{1}{3}\ln\lvert 3x+1\rvert-\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{x}{\sqrt{3}}+c$	Modulus must be included in the ln expression.
(c)	$\displaystyle\int_{\pi}^{2\pi}\sin^2\left(\frac{x}{8}\right)\mathrm{d}x$ $\displaystyle=\int_{\pi}^{2\pi}\frac{1}{2}\left[1-\cos\left(\frac{x}{4}\right)\right]\mathrm{d}x$ $=\left[\dfrac{1}{2}x-2\sin\left(\dfrac{x}{4}\right)\right]_{\pi}^{2\pi}$ $=\dfrac{\pi}{2}-2+\sqrt{2}$	Use of double angle formula $\cos 2A = 1 - 2\sin^2 A$ Use of special angle to find exact value.