H2 Mathematics (9758)
Chapter 6B 3D Vector Geometry
Assignment 2 (Lines & Planes) Solutions

1 2018/MJC Promo/I/7

The line l and plane Π_1 have equations

$$\mathbf{r}=\begin{pmatrix}2\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\1\end{pmatrix},\quad \lambda\in\mathbb{R}\quad \text{and}\quad x-2y+5z=-1$$

respectively.

(i) Find the acute angle between l and Π_1. [2]

(ii) Find the coordinates of *P*, the point of intersection of l and Π_1. [3]

(iii) Find the shortest distance from point $(2,1,-1)$ to Π_1. [3]

The plane Π_2 has equation

$$x+ky+2z=1,$$

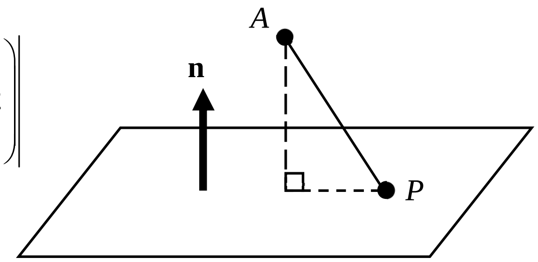
where *k* is a real constant.

(iv) Give a reason why Π_1 and Π_2 intersect in a line. [1]

(v) Given that *L* is the line where Π_1 and Π_2 meet, explain why point *P* lies on *L*. Hence or otherwise, find in terms of *k*, a vector equation of *L*. [4]

(vi) The plane Π_3 has equation $x+2y+3z=4$. Find the equation of the line of intersection between Π_1 and Π_3. [1]

1	Solution
(i)	Let θ be the acute angle between l and Π_1. $$\sin\theta = \frac{\left|\begin{pmatrix}1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\5\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-3\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\-2\\5\end{pmatrix}\right|}$$ $$= \frac{12}{\sqrt{11}\sqrt{30}}$$ $\theta = 41.344^\circ = 41.3^\circ$ (1 d.p.) or 0.722 rad (3 s.f)
(ii)	$$\Pi_1 : \mathbf{r}\bullet\begin{pmatrix}1\\-2\\5\end{pmatrix} = -1$$ $\overrightarrow{OP} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-3\\1\end{pmatrix} = \begin{pmatrix}2+\lambda\\1-3\lambda\\-1+\lambda\end{pmatrix}$ for some $\lambda \in \mathbb{R}$. $$\overrightarrow{OP}\bullet\begin{pmatrix}1\\-2\\5\end{pmatrix} = -1$$ To find point of intersection between line and plane, substitute the equation of the line into the equation of plane $$\begin{pmatrix}2+\lambda\\1-3\lambda\\-1+\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\5\end{pmatrix} = -1$$ $$2+\lambda-2+6\lambda-5+5\lambda = -1$$ $$12\lambda = 4$$ $$\lambda = \frac{1}{3}$$ $$\overrightarrow{OP} = \begin{pmatrix}\frac{7}{3}\\0\\-\frac{2}{3}\end{pmatrix}$$ $$P\left(\frac{7}{3},0,-\frac{2}{3}\right)$$
(iii)	$$\overrightarrow{AP} = \begin{pmatrix}\frac{7}{3}\\0\\-\frac{2}{3}\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\-1\\\frac{1}{3}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\-3\\1\end{pmatrix}$$

	Method 1: Shortest distance $= \left\|\overrightarrow{AP}\cdot\hat{\mathbf{n}}\right\|$ $= \left\|\frac{1}{3}\begin{pmatrix}1\\-3\\1\end{pmatrix}\cdot\frac{1}{\sqrt{30}}\begin{pmatrix}1\\-2\\5\end{pmatrix}\right\|$ $= \frac{12}{3\sqrt{30}}$ $= \frac{4}{\sqrt{30}}$ $= \frac{2\sqrt{30}}{15}$ or 0.730  **Method 2:** $\frac{\text{Shortest distance}}{AP} = \sin\theta$ Shortest distance $= AP\sin\theta$ $= \left\|\frac{1}{3}\begin{pmatrix}1\\-3\\1\end{pmatrix}\right\|\sin\left(41.344^\circ\right)$ $= \frac{1}{3}\sqrt{11}\sin\left(41.344^\circ\right)$ $= 0.730$
(iv)	$\mathbf{n}_1 = \begin{pmatrix}1\\-2\\5\end{pmatrix}$, $\mathbf{n}_2 = \begin{pmatrix}1\\k\\2\end{pmatrix} \Rightarrow \quad \mathbf{n}_1 \neq \alpha\mathbf{n}_2 \quad \alpha \in \mathbb{R}$ Since the normal vectors of both planes are not parallel, they are not parallel planes or equal planes. Hence they must intersect in a line.
(v)	$\Pi_2 : \mathbf{r}\cdot\begin{pmatrix}1\\k\\2\end{pmatrix} = 1$ $\overrightarrow{OP}\cdot\begin{pmatrix}1\\k\\2\end{pmatrix} = \begin{pmatrix}\frac{7}{3}\\0\\-\frac{2}{3}\end{pmatrix}\cdot\begin{pmatrix}1\\k\\2\end{pmatrix}$ $= \frac{7}{3} - \frac{4}{3}$ $= 1$

	Since P satisfies the equation of Π_2, it lies on Π_2. Since P lies on Π_1 and Π_2, it will lie on L. $\Pi_1 : \mathbf{r}\bullet\begin{pmatrix}1\\-2\\5\end{pmatrix}=-1 \qquad \Pi_2 : \mathbf{r}\bullet\begin{pmatrix}1\\k\\2\end{pmatrix}=1$ $\mathbf{n}_1\times\mathbf{n}_2=\begin{pmatrix}1\\-2\\5\end{pmatrix}\times\begin{pmatrix}1\\k\\2\end{pmatrix}$ $=\begin{pmatrix}-4-5k\\3\\k+2\end{pmatrix}$ $L:\mathbf{r}=\frac{1}{3}\begin{pmatrix}7\\0\\-2\end{pmatrix}+\mu\begin{pmatrix}-4-5k\\3\\k+2\end{pmatrix},\ \mu\in\mathbb{R}$
(vi)	Using GC, the equation of the line of intersection between Π_1 and Π_3 is $\mathbf{r}=\begin{pmatrix}\frac{3}{2}\\\frac{5}{4}\\0\end{pmatrix}+\gamma\begin{pmatrix}-4\\\frac{1}{2}\\1\end{pmatrix},\ \gamma\in\mathbb{R}$.

2 2018/DHS Prelim/I/9

The line l_1 passes through the point A with the position vector $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and is parallel to $\begin{pmatrix}t\\t^2+1\\3\end{pmatrix}$, while the cartesian equation of the plane p is given by $tx-2y+z=-3$, where t is a real constant. It is known that l_1 and p have no point in common.

(i) Show that $t=-1$. [3]

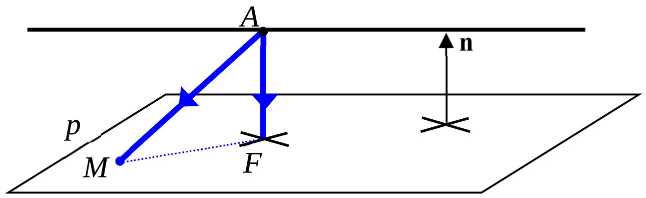
(ii) Find the distance between l_1 and p. [2]

(iii) The line l_2 has the cartesian equation $2y=z,\ x=3$. Show that l_2 lies on p. [2]

(iv) Given that point B and point C lie on l_1 and l_2 respectively, find $\overrightarrow{BC}$ such that it is perpendicular to both l_1 and l_2. [3]

(v) Find the vector equation of the line of reflection of l_1 in p. [3]

2	Solution
(i)	l_1: $\mathbf{r}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}t\\t^2+1\\3\end{pmatrix},\ \lambda\in\mathbb{R}$ $p: tx-2y+z=-3 \Rightarrow \mathbf{r}\bullet\begin{pmatrix}t\\-2\\1\end{pmatrix}=-3$ Since l_1 and p have no point in common, l_1 is parallel to p Since l_1 and p have no point in common, the direction vector of l_1 is perpendicular to normal vector of p and the point A does not lie on p. $\begin{pmatrix}t\\t^2+1\\3\end{pmatrix}\bullet\begin{pmatrix}t\\-2\\1\end{pmatrix}=0 \Rightarrow t^2-2t^2-2+3=0 \Rightarrow t=-1 \text{ or } 1$ $\begin{pmatrix}1\\3\\2\end{pmatrix}\bullet\begin{pmatrix}t\\-2\\1\end{pmatrix}\neq-3 \Rightarrow t-6+2\neq-3 \Rightarrow t\neq1$ Therefore, $t=-1$ (shown)

(ii)	The distance between l_1 and p = shortest distance of A to p $p: -x-2y+z=-3 \Rightarrow \mathbf{r}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$  Note that point $M(3,0,0)$ lies in p, since $\begin{pmatrix}3\\0\\0\end{pmatrix}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$ $\overrightarrow{AM}=\overrightarrow{OM}-\overrightarrow{OA}=\begin{pmatrix}3\\0\\0\end{pmatrix}-\begin{pmatrix}1\\3\\2\end{pmatrix}=\begin{pmatrix}2\\-3\\-2\end{pmatrix}$ Next, observe that finding the shortest distance from A to the plane p which is $\left\|\overrightarrow{AF}\right\|$ is same as finding the length of projection of $\overrightarrow{AM}$ onto $\mathbf{n}$. Shortest distance from A to the plane p $=\left\|\overrightarrow{AM}\bullet\hat{\mathbf{n}}\right\|$ $=\dfrac{\left\|\begin{pmatrix}2\\-3\\-2\end{pmatrix}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-1\\-2\\1\end{pmatrix}\right\|}=\dfrac{\|-2+6-2\|}{\sqrt{6}}=\dfrac{2}{\sqrt{6}}=\dfrac{\sqrt{6}}{3}$ units
(iii)	$l_2:\ 2y=z,\ x=3,\ \Rightarrow\ \mathbf{r}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\2\end{pmatrix}, \mu\in\mathbb{R}$ **<u>Method 1</u>** $\begin{pmatrix}0\\1\\2\end{pmatrix}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=0-2+2=0$ Therefore, l_2 is parallel to p. For l_2 lies on p, 1) Direction vector of l_2 is perpendicular to normal vector of p 2) Given position vector of l_2 lies on p $\begin{pmatrix}3\\0\\0\end{pmatrix}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$ Therefore, l_2 lies on p (shown). **<u>Method 2</u>**

	$l_2:\ 2y=z,\ x=3,\ \Rightarrow\ \mathbf{r}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\2\end{pmatrix},\mu\in\mathbb{R}$ ---(1) $p:-x-2y+z=-3\ \Rightarrow\ \mathbf{r}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$ --- (2) Substitute (1) into (2) $\left[\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\2\end{pmatrix}\right]\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3-2y+2y=-3=\text{RHS of }\ p$ Therefore, l_2 lies on p (shown).
(iv)	**<u>Method 1</u>** $\overrightarrow{OB}=\begin{pmatrix}1-\lambda\\3+2\lambda\\2+3\lambda\end{pmatrix},$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{OC}=\begin{pmatrix}3\\\mu\\2\mu\end{pmatrix},$ for some $\mu\in\mathbb{R}$ $\overrightarrow{BC}=\begin{pmatrix}3\\\mu\\2\mu\end{pmatrix}-\begin{pmatrix}1-\lambda\\3+2\lambda\\2+3\lambda\end{pmatrix}=\begin{pmatrix}2+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}$ Since BC is perpendicular to both l_1 and l_2, $\overrightarrow{BC}\bullet\mathbf{d}_1=0$ and $\overrightarrow{BC}\bullet\mathbf{d}_2=0$ $\begin{pmatrix}2+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix}=0$ and $\begin{pmatrix}2+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\1\\2\end{pmatrix}=0$ $7\lambda-4\mu=-7\quad\cdots(1)$ $\qquad 8\lambda-5\mu=-7\quad\cdots(2)$ Solve (1) and (2): $\mu=-\frac{7}{3}$ and $\lambda=-\frac{7}{3}$ $\overrightarrow{OB}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\left(-\frac{7}{3}\right)\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}10/3\\-5/3\\-5\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}3\\\mu\\2\mu\end{pmatrix}=\begin{pmatrix}3\\-7/3\\-14/3\end{pmatrix}$ $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}3\\-7/3\\-14/3\end{pmatrix}-\begin{pmatrix}10/3\\-5/3\\-5\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1\\-2\\1\end{pmatrix}$

Method 2

$$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Note: Since $\overrightarrow{BC}$ is perpendicular to l_1 and l_2, $\overrightarrow{BC}$ is parallel to $\mathbf{d}_1 \times \mathbf{d}_2$

$$\overrightarrow{BC} = \begin{pmatrix} 3-1+\lambda \\ \mu-3-2\lambda \\ 2\mu-2-3\lambda \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ \mu-3-2\lambda \\ 2\mu-2-3\lambda \end{pmatrix} = \beta \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Solving the equation, $\beta = \frac{1}{3}, \mu = \lambda = -\frac{7}{3}$

Hence $\overrightarrow{BC} = \frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

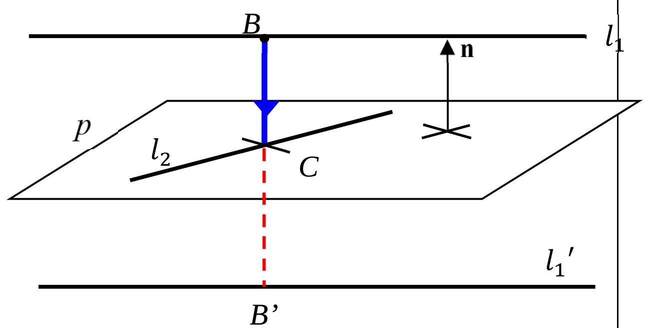
(v)

Method 1

Let l_1' be the line of reflection of l_1 in p.

Let point B' be the point lying on l_1' and is also the reflection of B in p.

$\mathbf{d}_{l_1'} = \mathbf{d}_{l_1} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ because l_1 is parallel to p.

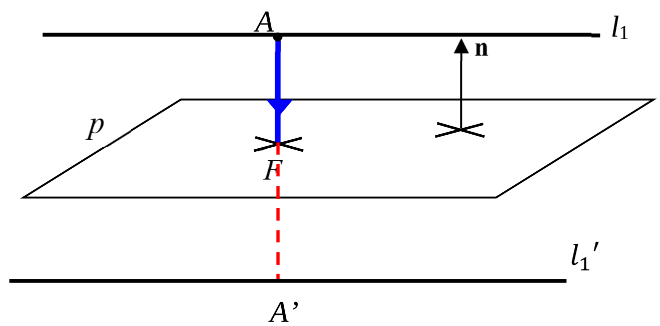


$$\overrightarrow{BB'} = 2\overrightarrow{BC} \Rightarrow \overrightarrow{OB'} = 2\overrightarrow{BC} + \overrightarrow{OB}$$

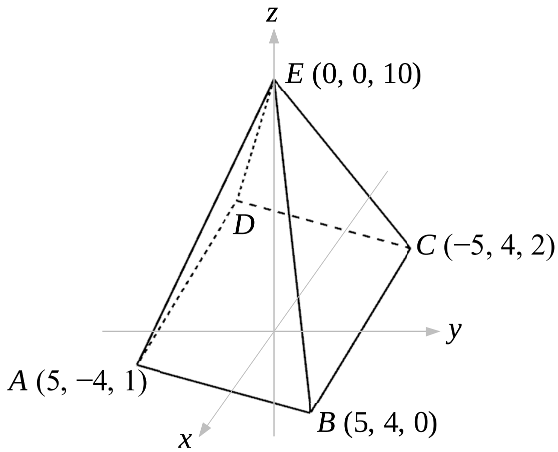
$$= 2\left[\frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right] + \begin{pmatrix} 10/3 \\ -5/3 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 8/3 \\ -3 \\ -13/3 \end{pmatrix}$$

Equation of line of reflection $l_1' : \mathbf{r} = \begin{pmatrix} 8/3 \\ -3 \\ -13/3 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \ \gamma \in \mathbb{R}$

	Method 2 Let point F be the foot of perpendicular of point A on plane p. $\overrightarrow{OF}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$  Since point F lies on plane p, $\begin{pmatrix}1-\lambda\\3-2\lambda\\2+\lambda\end{pmatrix}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$ Solving, $\lambda=\frac{1}{3}$ $\overrightarrow{OF}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\left(\frac{1}{3}\right)\begin{pmatrix}-1\\-2\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2\\7\\7\end{pmatrix}$ $\overrightarrow{OF}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}\Rightarrow\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$ $\overrightarrow{OA'}=\frac{2}{3}\begin{pmatrix}2\\7\\7\end{pmatrix}-\begin{pmatrix}1\\3\\2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1\\5\\8\end{pmatrix}$ Equation of line of reflection $l_1':\mathbf{r}=\begin{pmatrix}1/3\\5/3\\8/3\end{pmatrix}+\gamma\begin{pmatrix}-1\\2\\3\end{pmatrix},\ \gamma\in\mathbb{R}$

3 2018(9758)/II/3



An oblique pyramid has a plane base *ABCD* in the shape of a parallelogram. The coordinates of *A*, *B* and *C* are (5, −4, 1), (5, 4, 0) and (−5, 4, 2) respectively. The apex of the pyramid is at *E* (0, 0, 10) (see diagram).

(i) Find the coordinates of *D*. [1]

(ii) Find the cartesian equation of face *BCE*. [3]

(iii) Find the angle between face *BCE* and the base of the pyramid. [3]

(iv) Find the shortest distance from the midpoint of edge *AD* to face *BCE*. [5]

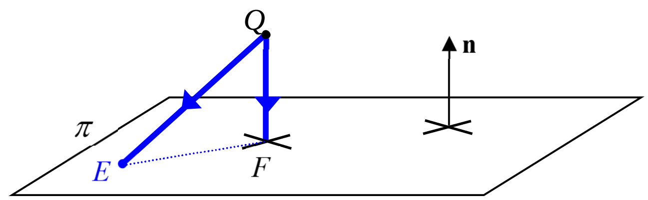
Q3	**Solution**
(i)	$\overrightarrow{AD} = \overrightarrow{BC}$ $\overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OB}$ $\overrightarrow{OD} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$ $\overrightarrow{OD} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$ Since *ABCD* forms a parallelogram, then $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are parallel and equal in length (and in the same direction). We can also say the same for $\overrightarrow{AB}$ and $\overrightarrow{DC}$. Therefore, coordinates of *D* are $(-5, -4, 3)$. Answer the question. Give your answer in coordinates form.

(ii)	$\overrightarrow{BE}=\overrightarrow{OE}-\overrightarrow{OB}=\begin{pmatrix}0\\0\\10\end{pmatrix}-\begin{pmatrix}5\\4\\0\end{pmatrix}=\begin{pmatrix}-5\\-4\\10\end{pmatrix}$ $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}-5\\4\\2\end{pmatrix}-\begin{pmatrix}5\\4\\0\end{pmatrix}=\begin{pmatrix}-10\\0\\2\end{pmatrix}$ $\overrightarrow{BE}\times\overrightarrow{BC}=\begin{pmatrix}-5\\-4\\10\end{pmatrix}\times\begin{pmatrix}-10\\0\\2\end{pmatrix}$ To get the normal vector of the place *BCE*, we cross 2 non-parallel vectors ($\overrightarrow{BE}$ and $\overrightarrow{BC}$) that are parallel to the plane. $=\begin{pmatrix}-8\\-90\\-40\end{pmatrix}=-2\begin{pmatrix}4\\45\\20\end{pmatrix}$ $\mathbf{r}\bullet\begin{pmatrix}4\\45\\20\end{pmatrix}=\begin{pmatrix}4\\45\\20\end{pmatrix}\bullet\begin{pmatrix}0\\0\\10\end{pmatrix}$ $=200$ $\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}4\\45\\20\end{pmatrix}=200$ Answer the question. Give your answer in **Cartesian form**. $4x+45y+20z=200$
(iii)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}5\\4\\0\end{pmatrix}-\begin{pmatrix}5\\-4\\1\end{pmatrix}=\begin{pmatrix}0\\8\\-1\end{pmatrix}$ $\overrightarrow{AB}\times\overrightarrow{BC}=\begin{pmatrix}0\\8\\-1\end{pmatrix}\times\begin{pmatrix}-10\\0\\2\end{pmatrix}$ To get the normal vector of the place *ABCD*, we cross 2 non-parallel vectors ($\overrightarrow{AB}$ and $\overrightarrow{BC}$) that are parallel to the plane. $=\begin{pmatrix}16\\10\\80\end{pmatrix}=2\begin{pmatrix}8\\5\\40\end{pmatrix}$ For **acute angle** (for between 2 planes, 2 lines, or a line and a plane), always input the **modulus**. Required angle $=\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}8\\5\\40\end{pmatrix}\bullet\begin{pmatrix}4\\45\\20\end{pmatrix}\right\rvert}{\sqrt{1689}\sqrt{2441}}$ $=58.6^\circ$ (1 d.p.) Always give **1 decimal place for angles in degree** unless exact or specified otherwise by question.

(iv) Let Q be the midpoint of AD

$$\overrightarrow{OQ}=\frac{\begin{pmatrix}5\\-4\\1\end{pmatrix}+\begin{pmatrix}-5\\-4\\3\end{pmatrix}}{2}=\begin{pmatrix}0\\-4\\2\end{pmatrix}$$

Apply Ratio Theorem or in this case, Midpoint Theorem.



Shortest distance $= QF$

$= \text{length of projection of } \overrightarrow{QE} \text{ onto } \mathbf{n}$

$$=\frac{\left|\overrightarrow{QE}\bullet\mathbf{n}\right|}{|\mathbf{n}|}$$

Remember your modulus.

$$=\frac{\left|\left(\begin{pmatrix}0\\0\\10\end{pmatrix}-\begin{pmatrix}0\\-4\\2\end{pmatrix}\right)\bullet\begin{pmatrix}4\\45\\20\end{pmatrix}\right|}{\sqrt{2441}}$$

$= 6.88 \quad (3 \text{ s.f.})$